Some Recent Results in Reverse Mathematics

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Outline

Introduction: Hilbert Program

- Hilbert Program: Justify "infinitary" math by "finitary" means.
- Program failed because of Gödel's Theorems. But...
- Motivating question: Let's find out the exact amount of "infinitary" tools needed.

Gödel Hierarchy

- Let T_1 and T_2 be theories. We say $T_1 < T_2$ iff T_2 proves the consistency of T_1.
- It turns out an almost linear hierarchy, quite robust (with some noise though).

Gödel Hierarchy

$$
\begin{array}{rl}
\textit{strong} & \left\{ \begin{array}{l} \vdots \\ \text{measurable cardinal} \\ \vdots \\ \text{ZFC} \\ \vdots \end{array} \right. \\[2ex]
\textit{medium} & \left\{ \begin{array}{l} \mathbb{Z}_2 \\ \vdots \\ \Pi^1_1\text{-CA} \\ \text{ATR}_0 \\ \text{ACA}_0 \end{array} \right. \\[2ex]
\textit{weak} & \left\{ \begin{array}{l} \text{WKL}_0 \\ \text{RCA}_0 \\ \vdots \\ \text{bounded arithmetic} \\ \vdots \end{array} \right.
\end{array}
$$

Goal of Reverse Mathematics

- Beginning of Reverse Mathematics: Harvey Friedman (1970's) and Steve Simpson.

- Goal: What set existence axioms are needed to prove the theorems of ordinary, classical (countable) mathematics?

Second Order Arithmetic $\mathbb{Z}_2$

- Two sorted language: (first order part) Numbers, $+$, $\times$; (second order part) Sets; $\in$.
- Most of "standard mathematics" can be done in $\mathbb{Z}_2$.
- This was initiated by Hilbert. (Set Theory is considered excessive.)

Subsystems of $\mathbb{Z}_2$ - The Big Five

Basic axioms and

- RCA_0: Σ_1-induction and Δ_0-comprehension for $\varphi \in \Delta_0$, $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.
- WKL_0: RCA_0 and every infinite binary tree has an infinite path.
- ACA_0: RCA_0 and for φ arithmetic, $\exists X \forall n (n \in X \leftrightarrow \varphi(n))$.
- ATR_0: RCA_0 and for every two well-orderings there is an isomorphism from one onto an initial segment of the other.
- Π^1_1-CA_0.

Remarks on Axioms

- They all assert the existence of certain sets.
- Some are measured by syntactical complexity.
- Some are from the analysis of mathematical tools.

RCA_0

- A set is called *decidable* or *recursive* or *computable* if there is an algorithm which decides its membership. E.g. the set of all prime numbers.
- In the (minimal) world RCA_0, only recursive sets exist.
- RCA_0 is the place to do constructive/finitary mathematics.

Examples from Classical Mathematics

Let the base theory be RCA_0

- WKL_0 is equivalent to Heine-Borel Theorem: Every open covering of $[0, 1]$ has a finite subcover.
- ACA_0 is equivalent to Bolzano-Weierstrass Theorem: Every bounded sequence of real numbers has a convergent subsequence.
- ATR_0 is equivalent to Perfect Set Theorem: Every uncountable closed set has a perfect subset.
- Π^1_1-CA_0 is equivalent to Cantor-Bendixson Theorem: Every closed subset of $\mathbb{R}$ is the union of a countable set and a perfect set.

A Remark on Philosophy

- RCA_0: Constructivism (Bishop)
- WKL_0: Finitistic reductions (Hilbert)
- ACA_0: Predicativism (Weyl, Feferman)
- ATR_0: Predicative reductionism (Friedman, Feferman)
- Π^1_1-CA_0: Impredicativeity (Feferman)

Recent Developments

- (old results) Simpson has a whole book about classical math theorems and their correspondence with big five.
- (Beyond the Big Five): Mummert and Simpson 2005 provide an example of reverse mathematics at the level of Π^1_2-CA. The results are in the area of general topology. (and H.Friedman's book draft.)
- More and more exceptions (chaos around Ramsey Theorem).

Frank Plumpton Ramsey Ramsey (1903 - 1930)

- Ramsey “was a British mathematician who, in addition to mathematics, made significant and precocious contributions in philosophy and economics before his death at the age of 26.”

Ramsey Theorem (History)

"One of the theorems proved by Ramsey in his 1930 paper *On a problem of formal logic* now bears his name (Ramsey's theorem). While this theorem is the work Ramsey is probably best remembered for, he only proved it in passing, as a minor lemma along the way to his true goal in the paper, solving a special case of the decision problem for first-order logic, ... A great amount of later work in mathematics was fruitfully developed out of the ostensibly minor lemma, which turned out to be an important early result in combinatorics, supporting the idea that within some sufficiently large systems, however disordered, there must be some order. So fruitful, in fact, was Ramsey's theorem that today there is an entire branch of mathematics, known as *Ramsey theory*, which is dedicated to studying similar results."

Ramsey Theorem

Definition

For $A \subseteq \mathbb{N}$, let $[A]^n$ denote the set of all n-element subsets of A.

Theorem (Ramsey, 1930)

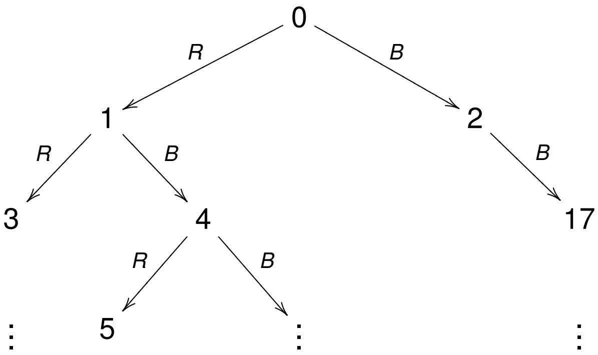
*Suppose $f : [\mathbb{N}]^n \to \{0, 1, \ldots, k-1\}$. Then there is an infinite set $H \subseteq \mathbb{N}$ which is f-*homogeneous*, i.e., f is constant on $[H]^n$.*

If we think of f as a k-coloring of the n-element subsets of $\mathbb{N}$, then all n-element subsets of H have the same color.

A Sketch of Proof of Ramsey Theorem for Pairs

Statement: If we colour pairs of natural numbers in two colors (Red and Blue), then there is an infinite subset $H \subset \mathbb{N}$, such that any pair formed by elements in H is coloured by the same colour.

Proof (idea). We enumerate a binary tree based on the colouring as illustrated by the following example:



Sketch of Proof (conti.)

We obtain an infinite binary tree.

So there is an infinite branch of the tree. (This fact is called Weak König Lemma.) For example, the path $0 \to 1 \to 4 \to 5 \to \cdots$.

We can then read from the branch a homogeneous set depending on whether we see infinitely many Red label or Blue label.

[Note: This tree is an r.e. tree, so this version of Weak König Lemma is stronger than WKL_0.]

Effective Ramsey Theorem

- Since 1972, Recursion Theorists also studied the "effective" content of Ramsey Theorem.
- Basic Question: Suppose f is recursive. How about the complexity of the homogeneous set H?
- Complexity could be measured by recursion theoretic or by reverse mathematical means.

Earlier Results

Theorem (Jockusch, 1972)

1. $\mathsf{ACA}_0 \vdash$ *(full version of) Ramsey Theorem.*
2. $\mathsf{RCA}_0 +$ "RT^3_2" *implies* ACA_0.
3. WKL_0 *doesn't imply Ramsey theorem for pairs.*

$$\mathsf{ACA}_0 \Leftrightarrow \mathsf{RT}^3_2 \Leftrightarrow \mathsf{RT}.$$

$$\mathsf{ACA}_0 \Rightarrow \mathsf{RT}^2_2 \quad \text{and} \quad \mathsf{WKL}_0 \not\Rightarrow \mathsf{RT}^2_2.$$

Another Earlier Result

Theorem (Seetapun and Slaman, 1995)

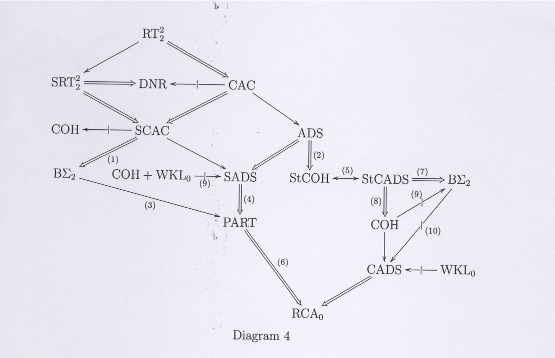
Ramsey Theorem for pairs is weaker than ACA_0.

In fact they proved a stronger result. It revived the area after more than 20 years silence.

$$\mathrm{ACA}_0 \Rightarrow \mathrm{RT}^2_2 \quad \text{and} \quad \mathrm{RT}^2_2 \not\Rightarrow \mathrm{ACA}_0.$$

Combinatorics below RT^2_2

Hirschfeldt and Shore [2007], *Combinatorial principles weaker than Ramsey's theorem for pairs*.



A New Breakthrough

- In 2010, Liu Jiayi, who was an undergraduate student in Zhong Nan University, showed that Ramsey Theorem for pairs does not prove WKL_0.

$$\mathsf{WKL}_0 \not\Rightarrow \mathsf{RT}^2_2 \quad \text{and} \quad \mathsf{RT}^2_2 \not\Rightarrow \mathsf{WKL}_0.$$

- (This was one of the most important questions in the area, and many recursion theorists have been working on it.)

Another New Result

- (Chong, Slaman and Yang 2012) Stable Ramsey Theorem for pairs does not imply Ramsey Theorem for Pairs.
- Nonstandard models are crucial in the proof. It is known that the method does not apply to ω.
- Question: How much induction should “finitary math” include?

References

1. Simpson, *Subsystems of Second-Order Arithmetic*, (second edition), ASL and CUP 2009.
2. Hirschfeldt and Shore, *Combinatorial principles weaker than Ramsey's theorem for pairs*, JSL, 2007.
3. Simpson, *The Gödel hierarchy and reverse mathematics*, Slides from Banff Conference, 2008
4. Shore, *Reverse mathematics: The playground of logic*, Slides from Gödel Lecture, Logic Colloquium '09, ASL, Sofia, 2009
5. Liu Jiayi, RT^2_2 *does not imply* WKL_0, to appear in JSL.