An Improved Theoretical Bound for Minimum CDS
in Wireless Ad Hoc Network

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Abstract

Since wireless ad hoc network lacks of physical network infrastructure, we can choose a connected dominating set (CDS) as the virtual backbone to improve the network performance. There exist many algorithms to find a minimum CDS in UDG and these algorithms usually include two phases. The first phase is to choose a maximal independent set (MIS) and the second phase is to connect them. In the performance analysis of those algorithms, the ratio of the size of selected MIS to the size of the optimal minimum CDS, which is also called the theoretical bound to approximate CDS, plays an important role to evaluate the effectiveness of the algorithms. Currently the best-known result of this ratio is $mis(G) \leq 3.4305mcds(G)+4.8185$. In this paper, we improved it by showing that $mis(G) \leq 3.3371mcds(G)+3.6741$.

Keywords: Connected Dominating Set; Maximal Independent Set; Unit Disk Graph

1 Introduction

A wireless ad hoc network consists of many nodes each of which is not only a mobile host but also a router. Owing to the own special properties, wireless ad hoc networks are often superior to traditional computer networks in many application areas, including traffic control, military applications, etc. However, such networks lack of physical infrastructures, resulting the difficulty to achieve scalability and efficiency. To solve the weakness, we usually select a virtual network infrastructure which is called virtual backbone to take the charge of routing for other nodes, such that the network topology will form a hierarchical structure and become more effective for network management.

To simplify our research, we assume that every node has the same transmission range. In this situation, we use Unit Disk Graph (UDG) to describe the network. An undirected graph $G = (V, E)$ is called a UDG only if every node has a broadcasting range 1. For two arbitrary vertices $v_1, v_2 \in V$, there exists an edge $(v_1, v_2) \in E$ only when the distance between v_1 and v_2 is at most 1. When constructing the virtual backbone, a connected dominating set (CDS) of the given graph is often our first choice. Given a graph $G = (V, E)$, a dominating set (DS) is a subset of V such that every vertex in the graph is either in this subset or adjacent to a vertex in the subset. Furthermore, a CDS is a connected DS, which means it can induce a connected subgraph. Obviously, the smaller size of a CDS results faster routing process and better network performance. However, Clark et.al. [1] proved that finding minimum CDS (MCDS) even in UDG is NP-hard. Thus lots of approximation algorithms for MCDS came out. These algorithms usually choose an maximal independent set (MIS) from UDG as a dominating set (DS) first, and then add some extra nodes to connect them. An MIS in a graph $G = (V, E)$ is a subset $M \subseteq V$ such that for arbitrary vertices $v_1, v_2 \in M$, the distance between v_1 and v_2 is more than 1, and if we insert a node u from $V \backslash M$ into M, M will not be an MIS any more. From the definition of MIS, we can easily prove that an MIS is also a DS. Therefore, the performance of the approximation algorithms highly depends on the relationship between the size of MIS and the size of MCDS in G. The approximation ratio $mis(G)/mcds(G)$ is called the theoretical bound to approximate CDS, where $mis(G)$ (respectively, $mcds(G)$) is the size of MIS (respectively, MCDS). Consequently, analyzing the theoretical bound is quite important to the analysis of such algorithms.

A lot of studies on the theoretical bound have been done before. Based on the fact that the neighborhood area of any node can contain at most five independent points, Wan [7] showed $mis(G) \leq 4mcds(G) + 1$. Wu [6] shown that the neighborhood area of any edge can contain at most eight independent points, and they gave a result as $mis(G) \leq 3.8mcds(G) + 1.2$. Along this direction, Wan [8] improved a better result that is $mis(G) \leq 3\frac{2}{3}mcds(G) + 1$. Later, Vahdatpour [9] claimed that $mis(G) \leq 3mcds(G) + 3$. However, the proof is far from a complete one. With approach of area and Voronoi division, Funke [2] showed that

$mis(G) \leq 3.453mcds(G) + 8.291$. However, the proof is also not complete and Gao [3] gave the detail proof. Recently, Li [4] improved the result into $mis(G) \leq 3.4305mcds(G) + 4.8185$.

In this paper, we will give a better theoretical bound to approximation CDS , which is $mis(G) \leq 3.3371mcds(G) + 3.6741$. To prove this bound, we first make our problem equivalent to a math problem of disk packing in Section 2. Then, we introduce the conception of effective area, and give the maximal condition that refers to the condition of packing maximal independent points in the neighborhood area of MCDS in Section 3. According to the maximal condition, we also did a lot of work on the relation between the length of boundary and the size of MIS in Section 4. Finally, Section 5 gives the conclusion and future works.

2 Preliminary

As mentioned in Section 1, calculating the theoretical bound of $mis(G)/mcds(G)$ has significant importance in the performance analysis of those two-phases approximation. Obviously, the key of this problem is trying to get the size of MIS and the size of MCDS in a UDG. Since finding the MCDS in a given UDG is NP-hard, we can estimate the size of MIS for a given MCDS. Let γ and α be the given size of MCDS and the size of MIS. It is easy to see that $\alpha = mis(G)$, while $\gamma = mcds(G)$.

Firstly, we define $disk_r(v)$ as the closed disk of radius r centered at v. Let $\Omega^{(1)} = \bigcup\limits_{o \in \text{MCDS}} disk_1(o)$. From the definition of MCDS, $\Omega^{(1)}$ represents the whole dominating area of the MCDS. All vertices in the corresponding UDG are located insides this area, including all vertices in any MIS. Therefore, our work is to find the MIS in $\Omega^{(1)}$. Considering the characteristic of MIS, for two arbitrary vertices $v_1, v_2 \in$ MIS, $disk_{0.5}(v_1)$ will never intersect with $disk_{0.5}(v_2)$. Also, $\forall v \in$ MIS, $disk_{0.5}(v)$ is always located insides $\Omega^{(1.5)} = \bigcup\limits_{o \in \text{MCDS}} disk_{1.5}(o)$. Consequently, calculating the possible number of vertices in MIS is equivalent to estimating the largest number of these disks located insides $\Omega^{(1.5)}$. Hence, our problem can be simplified as follows:

Problem: How many disjoint disks with $r = 0.5$ can locate within the area of $\Omega^{(1.5)}$? In addition, the centers of every disk must locate insides $\Omega^{(1)}$.

3 Maximal Condition

To place more disks with $r = 0.5$ in the area of $\Omega^{(1.5)}$, the disks should be arranged in the tightest way. It is easy to know that for two disks, the tightest arrangement is to make them tangential. For three disks, the tightest arranged way is to make every two disks tangential as shown in Fig. 1.

Routine 1 *If we ignore the boundary of $\Omega^{(1.5)}$, we can construct the tightest arrangement as follows:*

(1) Choose an arbitrary vertex $v_0 \in$ MIS, and define $\Omega = disk_{0.5}(v_0)$.

(2) Add a new vertex v_1 , and make $disk_{0.5}(v_1)$ tangential with $disk_{0.5}(v_0)$. $\Omega = \Omega \bigcup disk_{0.5}(v_1)$.

(3) Add a new vertex v_i and make $disk_{0.5}(v_i)$ tangential with two adjacent disks in Ω. $\Omega = \Omega \bigcup disk_{0.5}(v_i)$.

(4) Repeat step (3).
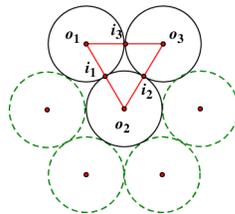
Figure 1: Example of tightest arrangement.

From Fig. 1 we find that there will always be some gaps among disks even in the tightest arrangement. Let s_g^0 denote the gap area in Fig. 1, then it is easy to see that

$$s_g^0 = area(i_1 i_2 i_3) = \frac{\sqrt{3}}{4} - \frac{\pi}{8}$$

Because of the existence of gaps, applying $\dfrac{area(\Omega^{(1.5)})}{area(disk_{0.5}(v))}$ to estimate α is a really rough method. To make a better result, we should take the effect of those gaps into account. That is to say, we need to make such disks share the gaps in some way. Obviously, Voronoi division is a nice choice to reach our purpose, but we will use another way in this paper.

3.1 Effective Area

According Fig. 1, the gap $area(i_1 i_2 i_3)$ should belong to three disks around it. Hence, each disk owns one third of the gap. Moreover, we can easily notice that $\angle o_1 o_2 o_3 = \frac{\pi}{3}$. Thus, $disk_{0.5}(o_2)$ can have 6 $\left(\frac{2\pi}{\frac{\pi}{3}}\right)$ gaps around. As the result, $disk_{0.5}(o_2)$ owns 2 on average. In order to simplify the description, we can introduce the conception of effective area.

Definition 1 (Effective Area) *For any gap formed in the process of placing disks in $\Omega^{(1.5)}$, it will be uniformly shared by the disks around it. Then the effective area of $disk_{0.5}(v)$ is*

$$s_v^{eff} = \frac{\pi}{4} + \sum_i \frac{s_g^i}{p_i},$$

where $\frac{\pi}{4}$ is the area of $disk_{0.5}(v)$, i refers to the number of gaps around $disk_{0.5}(v)$, s_g^i means the area of the i-th gap, and p_i is the number of disks around the i-th gap.

It is easy to find that if $\Omega^{(1.5)}$ is constructed by Routine 1, the effective area of a $disk_{0.5}(v)$ is

$$\frac{\pi}{4} + \sum_i \frac{s_g^i}{p_i} = \frac{\pi}{4} + 6 \times \frac{s_g^0}{3} = \frac{\sqrt{3}}{2}.$$

And we denote it with $s_{(1)}^{eff}$.

3.2 Considering the Boundary

All the discussion above ignores the boundary of $\Omega^{(1.5)}$. When we take the boundary into account, Routine 1 will not work anymore, because it is only used to guarantee the tightest arrangement of disks in the inner layer, not the outmost layer. To estimate the maximal number of disks in $\Omega^{(1.5)}$, we can divide the process into two steps:

(1) Make the outmost layer of disks tightest arranged. As all centers of those disks are located insides $\Omega^{(1)}$, we can make all the centers of disks laying in outmost layer located on the boundary of $\Omega^{(1)}$, as shown in Fig. 2.
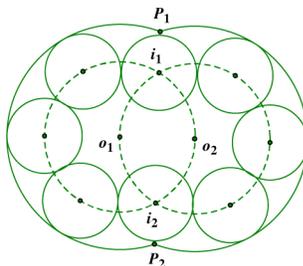

Figure 2: An example of making the outmost layer of disks tightest arranged.

(2) Deal with the inner space of $\Omega^{(1.5)}$ according to Routine 1. To avoid the shortcoming of Routine 1, we can add an extra condition that the result can be non-integer. That is to say, we can simply get the result by $\dfrac{\sum s^{eff}}{\sqrt{3}/2}$.

Then we can get the maximal condition.

Routine 2 *The condition to find an upper bound of the size of MIS can be as follows:*

(1) Make the boundary of $\Omega^{(1)}$ placed as many vertices that belong to MIS as possible.

(2) Apply $\dfrac{\sum s^{eff}}{\sqrt{3}/2}$ to calculate the number of inner vertices that belong to MIS.

Note: From Routine 2, what we get is an upper bound of the size of MIS, and we don't care whether the result can be reached in reality.

4   Main Results

From Routine 2, we can partition MIS into two subsets I_1 and I_2 defined by

$$
\begin{aligned}
I_1 &= \text{MIS}\backslash I_2, \\
I_2 &= o \in \text{MIS} : o \text{ is located on the boundary of } \Omega^{(1)}.
\end{aligned}
$$

Let α_1 and α_2 be the size of I_1 and I_2 respectively. The next lemma gives the maximal value of α_2.

Lemma 1 *The length of $\partial\Omega^{(1)}$ is at least $\frac{\pi}{3} \times \alpha_2$, where $\partial\Omega^{(1)}$ refers to the boundary of $\Omega^{(1)}$.*

Proof: After analyzing the boundary of $\Omega^{(1)}$, we can find there are only two kinds of elements on the boundary. They are arc whose radius is 1 and point of junction that connect two arcs. In order to mark different position on a circle, we can build polar coordinates on each disk as Fig. 3 showing. Let φ denote the different position and the value of φ increasing along counterclockwise direction and decreasing along clockwise direction. When a point walks along the boundary of $\Omega^{(1)}$, we can give it a property φ denoting the position on the circle where the point is being on. In the process of walking, the value of φ changes continuously on arcs while jumpily on points of junction. From Fig. 3 we can easily see that $\Delta\varphi$ on points of junction is equal to θ. And the value of θ is related to $dist(o_1, o_2)$. And the value of $\Delta\varphi$ is negative if the point walks along the boundary in negative direction. The definition of direction of boundary is as follow:

Definition 2 (The Direction of Boundary) *When a point walks along a boundary in a direction, the direction is called the positive direction (negative direction) if the area formed by the boundary is always on its left (right) side.*
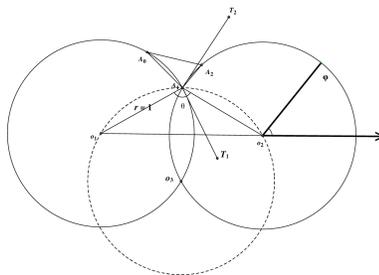
Figure 3: An example of boundary formed by two disks with $dist(o_1, o_2) \geq 1$

All in all, to make the size of MIS largest, the distance between any two adjacent vertices in MCDS is needed to be equal to 1. In this situation, $dist(o_1, o_2) \geq 1$ and θ will be at least $\frac{\pi}{3}$.

By Routine 2, we can let $dist(v_1, v_2) = 1$ for any two adjacent vertices $v_1, v_2 \in I_2$. To prove Lemma 1, we first prove that the length of the shorter path between v_1 and v_2 on the boundary is at least $\frac{\pi}{3}$. There are two kinds of situation to discuss.

(1) If v_1 and v_2 are on a same circle, then the length of the arc between v_1 and v_2 (denote it by $|arc(v_1, v_2)|$) is equal to $\frac{\pi}{3}$.

(2) When v_1 and v_2 are located on different circles, there will have several points of junction on the path between v_1 and v_2. Let q denote the number of those points. In this situation, we can first calculate $dist(v_1, v_2')$ with the length of the path between v_1 and v_2' being $\frac{\pi}{3}$.

(i) If $q = 1$, as in Fig. 3, the path between v_1 and v_2' is $arc(A_0 A_1) + arc(A_1 A_2)$. Hence,

$$\left| arc(v_1, v_2') \right| = |arc(A_0 A_1) + arc(A_1 A_2)| = \frac{\pi}{3}.$$

The Euclidean distance between v_1 and v_2' is $\left| \overrightarrow{A_0 A_2} \right|$, and $\left| \overrightarrow{A_0 A_2} \right| = \left| \overrightarrow{A_0 A_1} + \overrightarrow{A_1 A_2} \right|$. As radiuses of $arc(A_0 A_1)$ and $arc(A_1 A_2)$ are the same and equal to 1, we can move them to a same circle with $r = 1$ as Fig. 4 showing. Comparing Fig. 4 with Fig. 3, we can conclude that $\angle A_1' o A_1 = \theta$ and $\left| arc(A_0 A_1) + arc(A_1' A_2) \right| = \frac{\pi}{3}$. As $\theta > \frac{\pi}{3}$, there will be some free space between A_2 and A_0. Because of the translation invariance of vector,

$$dist(v_1, v_2') = \left| \overrightarrow{A_0 A_1} + \overrightarrow{A_1' A_2} \right| = \left| \overrightarrow{A_0 A_1} + \overrightarrow{B A_0} \right| = \left| \overrightarrow{B A_1} \right|,$$

where $\overrightarrow{B A_0} // \overrightarrow{A_1' A_2}$. On the circle, we can find point C making $|arc(C A_0)| = \left| arc(A_1' A_2) \right|$. Thus, $|arc(C A_1)| = \frac{\pi}{3}$ and $\left| \overrightarrow{C A_1} \right| = 1$ .In $\Delta A_0 C A_1$ and $\Delta A_0 B A_1$, $\left| \overrightarrow{C A_0} \right| = \left| \overrightarrow{B A_0} \right|$, $\angle A_1 A_0 C > \angle A_1 A_0 B$. According to law of cosines in triangle, it is easy to know

$$\left| \overrightarrow{B A_1} \right| < \left| \overrightarrow{C A_1} \right| = 1.$$

Then, $dist(v_1, v_2') < 1$. Because $dist(v_1, v_2) = 1$, v_2' is located on the path from v_1 to v_2. Therefore, $|arc(v_1, v_2)| > \left| arc(v_1, v_2') \right| = \frac{\pi}{3}$.

(ii) For $q > 1$, it is similar with case (i).

From situation (1) and (2) above, we can conclude that $|arc(v_1, v_2)|$ is always at least $\frac{\pi}{3}$. Then the value of α_2 is at most $\frac{|\partial \Omega^{(1)}|}{\pi/3}$ and $|\partial \Omega^{(1)}| \geq \frac{\pi}{3} \alpha_2$. Thus, Lemma 1 follows.
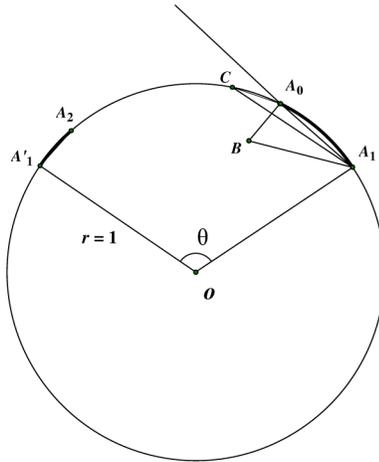
Figure 4: The model to calculate $|arc(v_1, v_2)|$

Theorem 1 *Let α and γ denote the size of MIS and the size of MCDS in a UDG. Then*

$$\alpha \leq 3.3371 \gamma + 3.6741$$

Proof: Firstly, it is easy to know that the length of $\Omega^{(1)}$ is at most $2\pi + \dfrac{2\pi}{3}(\gamma - 1)$. Considering Lemma 1, we can get the maximal value of α_2:

$$\alpha_2 \leq \frac{2\pi + \frac{2\pi}{3}(\gamma - 1)}{\frac{\pi}{3}} = 2\gamma + 4. \tag{1}$$

From Fig. 2 we can see that for $v \in I_2$, it is very complex to calculate s_v^{eff}. However, it is relatively easy to calculate $A_v^{eff} \bigcap \Omega^{(1)}$ (A_v^{eff} is the corresponding region of s_v^{eff}), which is shown in Fig. 5.
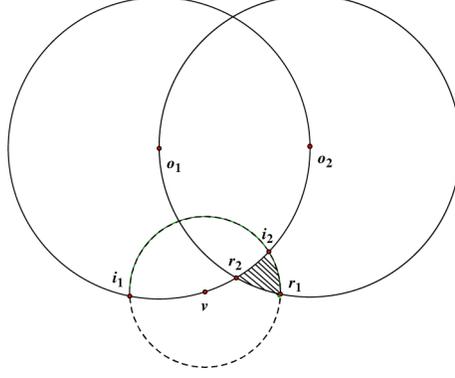
Figure 5: The minimal area of $disk_{0.5}(v) \bigcap \Omega^{(1)}$

When node v walks along the boundary of $\Omega^{(1)}$, the area of $disk_{0.5}(v) \bigcap \Omega^{(1)}$ has the minimal value that is equal to $area(disk_{0.5}(v) \bigcap disk_1(o_1))$. Also, there exist two neighbor disks around $disk_{0.5}(v)$ on the boundary of $\Omega^{(1)}$. Hence,

$$\left| A_v^{eff} \bigcap \Omega^{(1)} \right| \geq \frac{2}{3} s_g^0 + area(disk_{0.5}(v) \bigcap disk_1(o_1))$$

Next, we will calculate $area(disk_{0.5}(v) \bigcap disk_1(o_1))$.
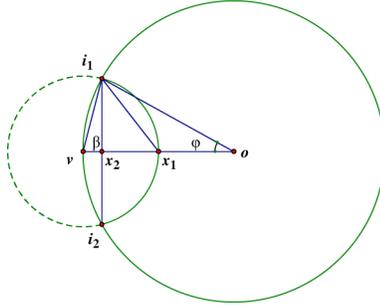
Figure 6: Calculating $\min area(disk_{0.5}(v) \bigcap disk_1(o))$

As shown in Fig. 6, we can get the result

$$area(disk_{0.5}(v) \bigcap disk_1(o)) = area(vi_2x_1i_1) = \varphi + \frac{\beta}{4} - \frac{1}{2}\sin 2\varphi - \frac{1}{8}\sin 2\beta,$$

where

$$\varphi = arcsin(\frac{\sqrt{15}}{8}); \qquad \beta = arcsin(\frac{\sqrt{15}}{4})$$

Hence,

$$\left| A_v^{eff} \bigcap \Omega^{(1)} \right| \geq \varphi + \frac{\beta}{4} - \frac{1}{2}\sin 2\varphi - \frac{1}{8}\sin 2\beta + \frac{\sqrt{3}}{6} - \frac{\pi}{12} = 0.3776 = s_{(2)}^{eff}. \tag{2}$$

By Routine 2,

$$area(\Omega^{(1)}) \geq \sum_{v \in I_2} A_v^{eff} \bigcap \Omega^{(1)} + \alpha_1 s_{(1)}^{eff}.$$

According to the equation (2),

$$area(\Omega^{(1)}) \geq \alpha_1 s_{(1)}^{eff} + \alpha_2 s_{(2)}^{eff} = \alpha s_{(1)}^{eff} + \alpha_2(s_{(2)}^{eff} - s_{(1)}^{eff}) \tag{3}$$

It is easy to prove by induction on that

$$area(\Omega^{(1)}) \leq \pi + (\frac{\pi}{3} + \frac{\sqrt{3}}{2})(\gamma - 1). \tag{4}$$

The inequalities (1), (3), (4) imply that

$$\alpha \leq \frac{\frac{\pi}{3} + \frac{3\sqrt{3}}{2} - 2s_{(2)}^{eff}}{s_{(1)}^{eff}}\gamma + \frac{\frac{2\pi}{3} + \frac{3\sqrt{3}}{2} - 4s_{(2)}^{eff}}{s_{(1)}^{eff}} \approx 3.3371\gamma + 3.6741.$$

Thus, Theorem 1 follows.

5   Conclusion

In this paper, we presented a better relation for MIS and MCDS in a UDG with more consideration of the boundary effect. We use the method of effective area to deal with packing problem in the dominating area of MCDS, and then replace the originally used hexagon with this new area to calculate a smaller ratio. The final result is $mis(G) \leq 3.3371mcds(G) + 3.6741$. To apply geometry to our problem, we assume the distance between two points in MIS is greater or equal to 1 and actually we let it equal to 1 to reach the closest packing. If we remove the condition of "equal to 1" above, the result will be better. Thus, we still have a lot of work to do to get the real "theoretical bound".

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