Lab05 - Matroid

Exercises for Algorithms by Xiaofeng Gao, 2016 Spring Semester

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1. Provide an example of $(S, \mathbf{C})$ which is an independent system but not a matroid. Give an instance of S such that $v(S) \neq u(S)$ (should be different from the example posted in class).

2. Matching matroid M_C: Let $G = (V, E)$ be an arbitrary undirected graph. $\mathbf{C}$ is the collection of all vertices set which can be covered by a matching in G.

   (a) Prove that $M_C = (V, \mathbf{C})$ is a matroid.

   (b) Given a graph $G = (V, E)$ where each vertex v_i has a weight $w(v_i)$, please give an algorithm to find the matching where the weight of all covered vertices is maximum. Prove the correctness and analyze the time complexity of your algorithm.

   Note: Given a graph $G = (V, E)$, a matching M in G is a set of pairwise non-adjacent edges; that is, no two edges share a common vertex. A vertex is covered (or matched) if it is an endpoint of one of the edges in the matching. Otherwise the vertex is uncovered.

3. A Dyck path of length $2n$ is a path in the plane from $(0, 0)$ to $(2n, 0)$, with steps $U = (1, 1)$ and $D = (1, -1)$, that never passes below the x-axis. For example, $P = UUDUDUUDDD$ is a Dyck path of length 10. Each Dyck path defines an up-step set: the subset of $[2n]$ consisting of the integers i such that the i-th step of the path is U. The up-step set of P is $\{1, 2, 4, 6, 7\}$. Let $\mathbf{B_n}$ be the collection of up-step sets of the Dyck paths of length $2k$ $(0 \leq k \leq n)$. Prove that $\mathbf{B_n}$ is the collection of bases of a matroid.

4. G be an undirected graph. A set of cycles $\{c_1, c_2, \ldots, c_k\}$ in G is called redundant if every edge in G appears in an even number of c_i's. A set of cycles is *independent* if it contains no redundant subset. A maximal independent set of cycles is called a *cycle basis* for G.

   (a) Let C be any cycle basis for G. Prove that for any cycle γ in G, there is a subset $A \subseteq C$ such that $A \cup \{\gamma\}$ is redundant. In other words, is the 'exclusive or' of the cycles in A.

   (b) Prove that the set of independent cycle sets form a matroid.

   (c) Now suppose each edge of G has a weight. Define the weight of a cycle to be the total weight of its edges, and the weight of a set of cycles to be the total weight of all cycles in the set. (Thus, each edge is counted once for every cycle in which it appears.) Describe and analyze an efficient algorithm to compute the minimum-weight cycle basis in G.