Matroid*

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X033533-Algorithm: Analysis and Theory

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Outline

1. Matroid
   - Independent System
   - Matroid
2. Greedy Algorithm on Matroid
   - $u(F)$ and $v(F)$
   - Greedy-MAX Algorithm
3. Task Scheduling Problem
   - Unit-Time Task Scheduling
   - Greedy Approach

Independent System

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We say that $\mathbf{C}$ is hereditary if it satisfies this property.

Each subset in $\mathbf{C}$ is called an independent subset.

Note that the empty set $\emptyset$ is necessarily a member of $\mathbf{C}$.

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Example: Given an undirected graph $G = (V, E)$, Define $\mathbf{H}$ as:

$\mathbf{H} = \{F \subseteq E \mid F$ is a Hamiltonian circuit or a union of disjoint paths$\}$.

Then $(E, \mathbf{H})$ is an independent system.

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Then $(E, \mathbf{H})$ is an independent system.

Proof: (Hereditary)

Given any $F \in \mathbf{H}$ and $P \subset F$.

Since F is either a Hamiltonian circuit or a union of disjoint path, P must be a union of disjoint paths, which obviously belongs to $\mathbf{H}$. $\square$

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Matroid

An independent system $(S, \mathbf{C})$ is a matroid if it satisfies the exchange property:

$$A, B \in \mathbf{C} \text{ and } |A| > |B| \Rightarrow \exists\, x \in A \backslash B \text{ such that } B \cup \{x\} \in \mathbf{C}.$$

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Thus a matroid should satisfy two requirements: **hereditary** and **exchange property**.

Matric Matroid

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Proof:

- Hereditary: If $A \subset B$ and $B \in \mathbf{C}$, meaning B is a linearly independent subset of row vectors of M, then A must be linearly independent.
- Exchange Property: The exchange property is a well known fact for linearly independence. Say, If A, B are sets of linearly independent rows of M, and $|A| < |B|$, then dim span$(A) <$ dim span(B). Choose a row x in B that is not contained in span(A). Then $A \cup \{x\}$ is a linearly independent subset of rows of M. $\square$

Graphic Matroid

Graphic Matroid M_G: Consider a (undirected) graph $G = (V, E)$. Let $S = E$ and $\mathbf{C}$ the collection of all edge sets each of which induces an acyclic subgraph of G. Then $M_G = (S, \mathbf{C})$ is a matroid.

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Proof:

- Hereditary: If B is an edge set which induces an acyclic subgraph of G, obviously any $A \subset B$ induces an acyclic subgraph.
- Exchange Property: consider $A, B \in \mathbf{C}$ with $|A| > |B|$.

  Note that (V, A) has $|V| - |A|$ connected components and (V, B) has $|V| - |B|$ connected components.

  Hence, A has an edge e connecting two connected components of (V, B), which implies $B \cup \{e\} \in \mathbf{C}$. $\square$

More Examples

Uniform matroid $U_{k,n}$: A subset $X \subseteq \{1, 2, \cdots, n\}$ is independent if and only if $|X| \leq k$.

Cographic matroid M_G^*: Let $G = (V, E)$ be an arbitrary undirected graph. A subset $I \subseteq E$ is independent if the complementary subgraph $(V, E \backslash I)$ of G is connected.

Matching matroid: Let $G = (V, E)$ be an arbitrary undirected graph. A subset $I \subseteq V$ is independent if there is a matching in G that covers I.

Disjoint path matroid: Let $G = (V, E)$ be an arbitrary directed graph, and let s be a fixed vertex of G. A subset $I \subseteq V$ is independent if and only if there are edge-disjoint paths from s to each vertex in I.

Notation

The word “matroid" is due to Hassler Whitney[1], who first studied matric matroid (1935).

Actually the greedy algorithm first appeared in the combinatorial optimization literature by Jack Edmonds[2] (1971).

An extension of matroid theory to **greedoid** theory was pioneered by Korte and Lovász, who greatly generalize the theory (1981-1984).

Hassler Whitney
(1907-1989)
Wolf Prize (1983)

[1] Hassler Whitney. On the abstract properties of linear dependence. *American Journal of Mathematics*, 57:509-533, 1935.

[2] Jack Edmonds. Matroids and the greedy algorithm. *Mathematical Programming*, 1:126-136, 1971.

Outline

Extension

An element x is called an extension of an independent subset I if $x \notin I$ and $I \cup \{x\}$ is independent.

An independent subset is maximal if it has no extension.

For any subset $F \subseteq S$, an independent subset $I \subseteq F$ is maximal in F if I has no extension in F.

Maximal Independent Subset

Consider an independent system $(S, \mathbf{C})$. For $F \subseteq S$, define

$$\begin{aligned} u(F) &= \min\{|I| \mid I \text{ is a maximal independent subset of } F\} \\ v(F) &= \max\{|I| \mid I \text{ is an independent subset of } F\} \end{aligned}$$

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Independent Vertex Set: Given a graph $G = (V, E)$, an independent vertex set is a subset $I \subseteq V$ such that any two vertices in I are not directly connected.

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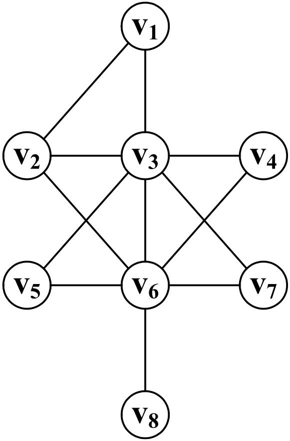
I is **maximum** if it is with the largest cardinality among all maximal independent vertex set. ($v(V)$ is the cardinality value of any maximum independent vertex set I.)

An Independent Vertex Set Instance



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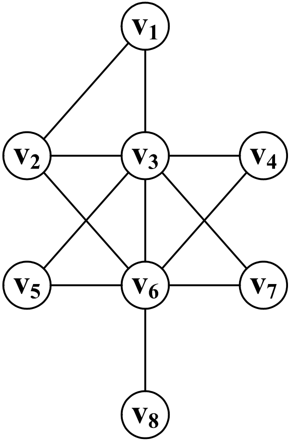

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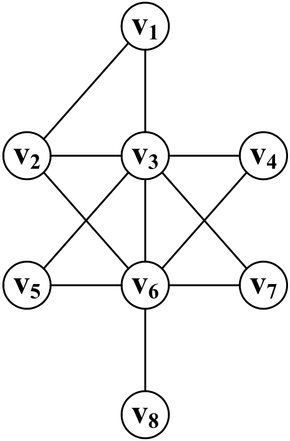

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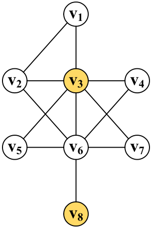

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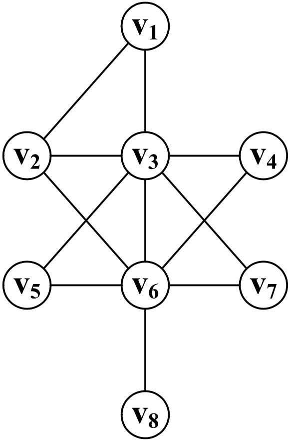

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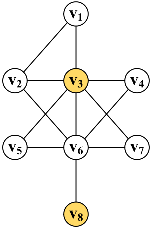

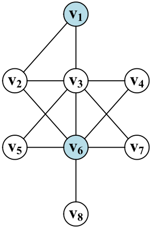

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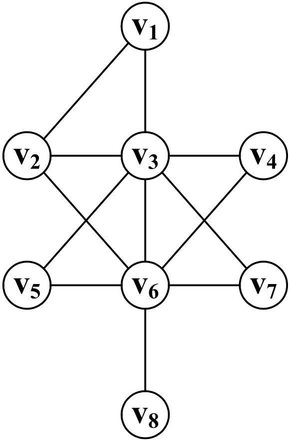

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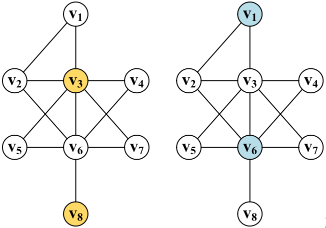

$u(V) = 2$

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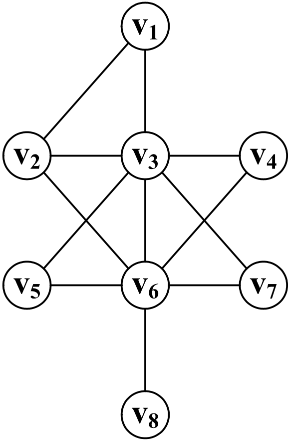

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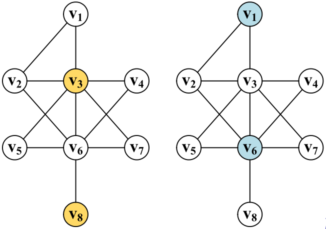

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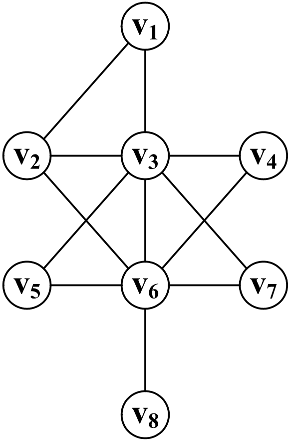

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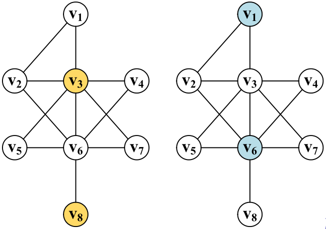

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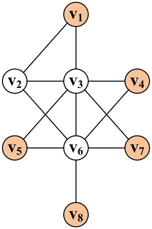

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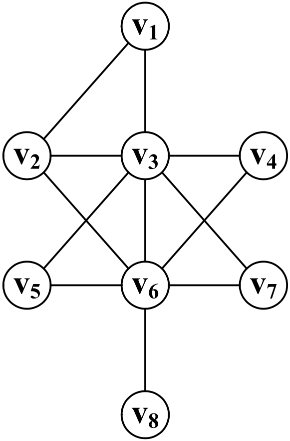

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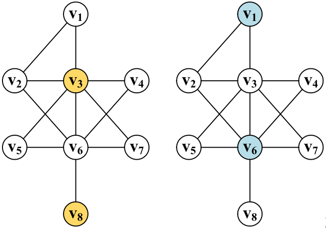

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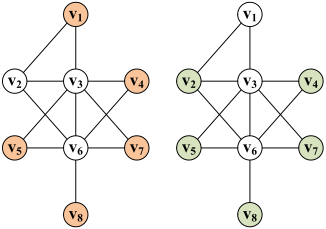

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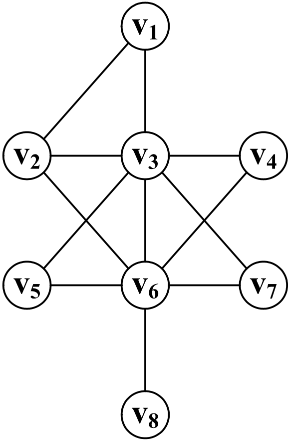

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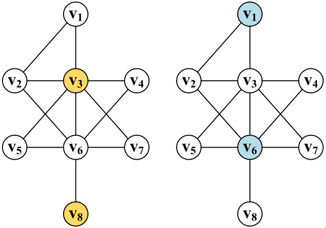

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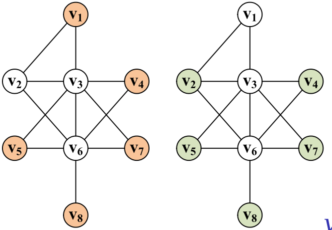

$$v(V) = 5$$

Matroid Theorem

Theorem: An independent system $(S, \mathbf{C})$ is a matroid if and only if for any $F \subseteq S$, $u(F) = v(F)$.

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($\Leftarrow$) Consider two independent subsets A and B with $|A| > |B|$. Set $F = A \cup B$. Then every maximal independent subset I of F has size $|I| \geq |A| > |B|$. Hence, B cannot be a maximal independent subset of F, so B has an extension in F. $\square$

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Thus the definition of matroid could be either by exchange property or by $u(F) = v(F)$.

Corollary

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Proof: (Contradiction) Suppose A and B are two maximal independent subsets with $|A| > |B|$, then B must have an extension in $A \cup B$, which violates its maximality property. $\square$

Outline

Basis

In a matroid $(S, \mathbf{C})$, every maximal independent subset of S is called a **basis** (some reference call it **base**).

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Example: In a graphic matroid $M_G = (S, \mathbf{C})$, $A \in \mathbf{C}$ is a basis if and only if A is a spanning tree.

Weighted Independent System

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In a weighted matroid, there is a maximum weight independent subset which is a basis.

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In a weighted matroid, there is a maximum weight independent subset which is a basis.

Note: we can define the associated strictly positive weight function $c(\cdot)$ to each element $x \in S$. Thus the weight function extends to subsets of S by summation:

$$c(A) = \sum_{x \in A} c(x).$$

Greedy Algorithm for Independent System

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The algorithm is written as:

Algorithm 1: Greedy-MAX

Sort all elements in S into ordering $c(x_1) \geq c(x_2) \geq \cdots \geq c(x_n)$;
$A \leftarrow \emptyset$;
for $i = 1$ *to* n **do**
    if $A \cup \{x_i\} \in \mathbf{C}$ **then**
        $A \leftarrow A \cup \{x_i\}$;
output A;

Time Complexity

Let $n = |S|$ = number elements in S. Then sorting the elements of S requires $O(n \log n)$.

The **for-loop** iterates n times. In the body of the loop one needs to check whether $A \cup \{x\}$ is in $\mathbf{C}$. If each check takes $f(n)$ time, then the loop takes $O(nf(n))$ time.

Thus, Greedy-MAX takes $O(n \log n + nf(n))$ time.

Greedy Theorem for Independent System

Theorem: Consider a weighted independent system. Let A_G be obtained by the Greedy Algorithm. Let A^* be an optimal solution. Then

$$1 \leq \frac{c(A^*)}{c(A_G)} \leq \max_{F \subseteq S} \frac{v(F)}{u(F)}$$

where $v(F)$ is the maximum size of independent subset in F and $u(F)$ is the minimum size of maximal independent subset in F.

Proof

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(By Contradiction) If not, there exists an element $x_j \in S_i \backslash A_G$ such that $(S_i \cap A_G) \cup \{x_j\}$ is independent.

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However, at the beginning of the jth iteration of the loop in the Greedy-Max, x_j must be selected into A_G^{j-1}. (Since $A_G^{j-1} \cup \{x_j\}$ must be a subset of $(S_j \cap A_G) \cup \{x_j\}$, and hence, is an independent set.)

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Therefore we have $|S_i \cap A_G| \geq u(S_i)$.

Moreover, since $S_i \cap A^*$ is independent, we have $|S_i \cap A^*| \leq v(S_i)$.

Proof (2)

Now we express $c(A_G)$ and $c(A^*)$ in terms of $|S_i \cap A_G|$ and $|S_i \cap A^*|$.

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Therefore,

$$\begin{aligned} c(A_G) &= \sum_{x_i \in A_G} c(x_i) \\ &= c(x_1) \cdot |S_1 \cap A_G| + \sum_{i=2}^{n} c(x_i) \cdot (|S_i \cap A_G| - |S_{i-1} \cap A_G|) \\ &= \sum_{i=1}^{n-1} |S_i \cap A_G| \cdot (c(x_i) - c(x_{i+1})) + |S_n \cap A_G| \cdot c(x_n) \end{aligned}$$

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Similarly,

$$c(A^*) = \sum_{i=1}^{n-1} |S_i \cap A^*| \cdot (c(x_i) - c(x_{i+1})) + |S_n \cap A^*| \cdot c(x_n)$$

Proof (3)

Define $\rho = \max_{F \subseteq S} \frac{v(F)}{u(F)}$. Then we have

$$\begin{aligned}
c(A^*) &= \sum_{i=1}^{n-1} |S_i \cap A^*| \cdot (c(x_i) - c(x_{i+1})) + |S_n \cap A^*| \cdot c(x_n) \\
&\leq \sum_{i=1}^{n-1} v(S_i) \cdot (c(x_i) - c(x_{i+1})) + v(S_n) \cdot c(x_n) \\
&\leq \sum_{i=1}^{n-1} \rho \cdot u(S_i) \cdot (c(x_i) - c(x_{i+1})) + \rho \cdot u(S_n) \cdot c(x_n) \\
&\leq \sum_{i=1}^{n-1} \rho \cdot |S_i \cap A_G| \cdot (c(x_i) - c(x_{i+1})) + \rho \cdot |S_n \cap A_G| \cdot c(x_n) \\
&= \rho \cdot c(A_G).
\end{aligned}$$

Proof (4)

Thus,

$$1 \le \frac{c(A^*)}{c(A_G)} \le \rho = \max_{F \subseteq S} \frac{v(F)}{u(F)}. \qquad \square$$

Note: This theorem implies that if we use Greedy-MAX to find a subset $I \in \mathbf{C}$ with the maximum weight, the result will not be that bad.

It is bounded by the size of the maximum size independent subset of S versus the minimum size maximal independent subset of S. Say,

$$\frac{1}{\rho} \cdot c(A^*) \le c(A_G) \le c(A^*).$$

Corollary for Matroid

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Proof: Since in a matroid for any $F \subseteq S$, $u(F) = v(F)$, the corollary can be directly derived from the previous theorem. $\square$

Minimizing or Maximizing?

Let $M = (S, \mathbf{C})$ be a matroid.

The algorithm Greedy-MAX(M, c) returns a set $I \in \mathbf{C}$ maximizing the weight $c(I)$.

If we would like to find a set $I \in \mathbf{C}$ with minimal weight, then we can use Greedy-MAX with weight function

$$c^*(x_i) = m - c(x_i), \qquad \forall x_i \in I,$$

where m is a real number such that $m > \max_{x_i \in S} c(x_i)$.

An Example: Graphic Matroid

Minimum Spanning Tree: For a connected graph $G = (V, E)$ with edge weight $c : E \to \mathbb{R}^+$, computing the minimum spanning tree.

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This is because every maximum weight independent set is a base, i.e., a spanning tree which contains a fixed number of edges.

$$c^*(A) = (|V| - 1)c_{\max} - c(A).$$

An independent subset that maximizes the quantity $c^*(A)$ must minimize $c(A)$.

An Example (Cont.)

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We could also use the property of Greedy-MAX on Matroid to validate the correctness of the Kruskal algorithm.

More Examples

Matric matroid: Given a matrix M, compute a subset of vectors of maximum total weight that span the column space of M.

Uniform matroid: Given a set of weighted objects, compute its k largest elements.

Cographic matroid: Given a graph with weighted edges, compute its minimum spanning tree.

Matching matroid: Given a graph, determine whether it has a perfect matching.

Disjoint path matroid: Given a directed graph with a special vertex s, find the largest set of edge-disjoint paths from s to other vertices.

Matroid v.s. Greedy-MAX

Theorem: An independent system $(S, \mathbf{C})$ is a matroid if and only if for any cost function $c(\cdot)$, the Greedy-MAX algorithm gives an optimal solution.

Matroid v.s. Greedy-MAX

Theorem: An independent system $(S, \mathbf{C})$ is a matroid if and only if for any cost function $c(\cdot)$, the Greedy-MAX algorithm gives an optimal solution.

Proof. ($\Rightarrow$) When $(S, \mathbf{C})$ is a matroid, $u(F) = v(F)$ for any $F \subseteq S$. Therefore, Greedy-MAX gives optimal solution.

Next, we show ($\Leftarrow$).

Sufficiency

($\Leftarrow$) For contradiction, suppose independent system $(S, \mathbf{C})$ is not a matroid. Then there exists $F \subseteq S$ such that F has two maximal independent sets I and J with $|I| < |J|$. Define

$$c(e) = \begin{cases} 1+\varepsilon & \text{if } e \in I \\ 1 & \text{if } e \in J \backslash I \\ 0 & \text{if } e \in S \backslash (I \cup J) \end{cases}$$

where ε is a sufficiently small positive number to satisfy $c(I) < c(J)$. Then the Greedy-MAX algorithm will produce I, which is not optimal! $\square$

Outline

Unit-Time Task

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For example, the first task in the schedule begins at time 0 and finishes at time 1, the second task begins at time 1 and finishes at time 2, and so on.

Unit-time Task Scheduling Problem

The problem of **scheduling unit-time tasks with deadlines and penalties for a single processor** has the following inputs:

- a set $S = \{1, 2, ..., n\}$ of n unit-time tasks;
- a set of n integer deadlines $d_1, d_2, \ldots, d_n$, such that each d_i satisfies $1 \leq d_i \leq n$ and task i is supposed to finish by time d_i;
- a set of n nonnegative weights or penalties $w_1, w_2, \ldots, w_n$, such that a penalty w_i is incurred if task i is not finished by time d_i.

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- a set of n nonnegative weights or penalties $w_1, w_2, \ldots, w_n$, such that a penalty w_i is incurred if task i is not finished by time d_i.

Requirement: find a schedule for S on a machine within time n that minimizes the total penalty incurred for missed deadline.

Properties of a Schedule

Given a schedule S, Define:

Early: a task is early in S if it finishes before its deadline.
Late: a task is late in S if it finishes after its deadline.

Early-First Form: S is in the early-first form if the early tasks precede the late tasks.

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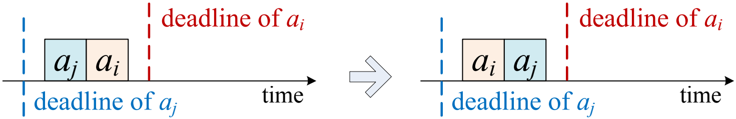
Claim: An arbitrary schedule can always be put into *early-first form* without changing its penalty value.



Properties of a Schedule (2)

Canonical Form: An arbitrary schedule can always be transformed into *canonical form*, in which the early tasks precede the late tasks and are scheduled in order of monotonically increasing deadlines.

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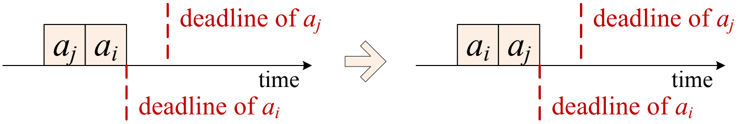
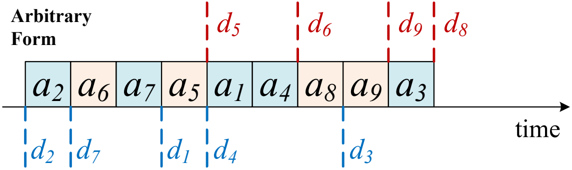
First put the schedule into early-first form.

Then swap the position of any consecutive early tasks a_i and a_j if $d_j > d_i$ but a_j appears before a_i.



An Example


Arbitrary Form
a_2 a_6 a_7 a_5 a_1 a_4 a_8 a_9 a_3
d_5 d_6 d_9 d_8
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time

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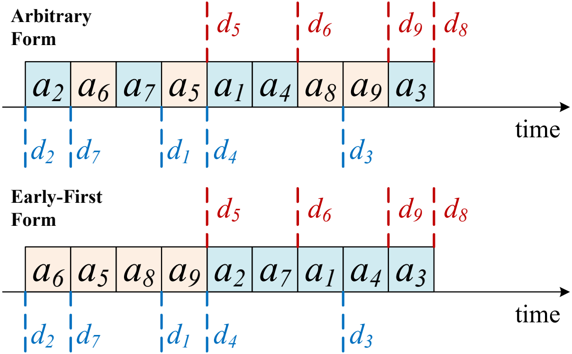
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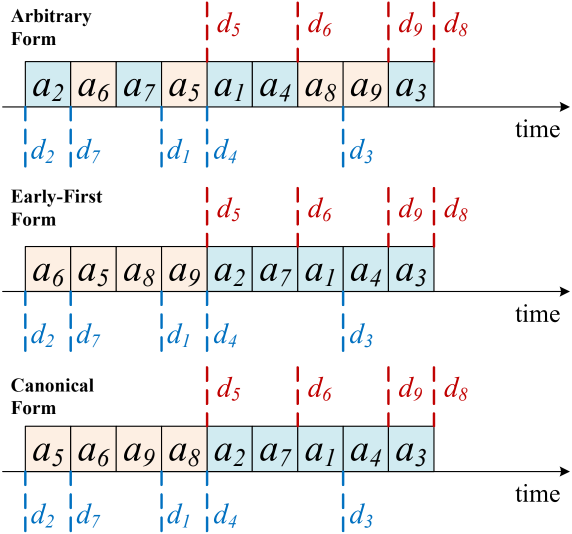
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Outline

1. Matroid
   - Independent System
   - Matroid
2. Greedy Algorithm on Matroid
   - $u(F)$ and $v(F)$
   - Greedy-MAX Algorithm
3. Task Scheduling Problem
   - Unit-Time Task Scheduling
   - Greedy Approach

Reduction

The search for an optimal schedule S thus reduces to finding a set A of tasks that we assign to be early in the optimal schedule.

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The search for an optimal schedule S thus reduces to finding a set A of tasks that we assign to be early in the optimal schedule.

To determine A, we can create the actual schedule by listing the elements of A in order of monotonically increasing deadlines, then listing the late tasks (i.e., $S - A$) in any order, producing a canonical ordering of the optimal schedule.

Independence

Independent: A set of tasks A is independent if there exists a schedule for these tasks without penalty.

Clearly, the set of early tasks for a schedule forms an independent set of tasks. Let **C** denote the set of all independent sets of tasks.

For $t = 0, 1, 2, \cdots, n$, let

$N_t(A)$ denote the number of tasks in A whose deadline is t or earlier.

Note that $N_0(A) = 0$ for any set A.

Lemma

Lemma: For any set of tasks A, the statements (1)-(3) are equivalent.
(1). The set A is independent.
(2). For $t = 0, 1, 2, \cdots, n$, $N_t(A) \leq t$.
(3). If the tasks in A are scheduled in order of monotonically increasing deadlines, then no task is late.

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Proof:
$\neg(2) \Rightarrow \neg(1)$: if $N_t(A) > t$ for some t, then there is no way to make a schedule with no late tasks for set A, because more than t tasks must finish before time t. Therefore, (1) implies (2).

$(2) \Rightarrow (3)$: there is no way to "get stuck" when scheduling the tasks in order of monotonically increasing deadlines, since (2) implies that the ith largest deadline is at least i.

$(3) \Rightarrow (1)$: trivial. $\square$

Greedy Approach

Use the previous lemma, we can easily compute whether or not a given set of tasks is independent.

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Thus if $(S, \mathbf{C})$ is a matroid, then we can use Greedy-MAX to find an independent set A of tasks with the maximum total penalty, which is proved to be an optimal solution.

Matroid Theorem

Theorem: Let S be a set of unit-time tasks with deadlines and $\mathbf{C}$ the set of all independent tasks of S. Then $(S, \mathbf{C})$ is a matroid.

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Proof: (Hereditary): Trivial.
(Exchange Property): Consider two independent sets A and B with $|A| < |B|$. Let k be the largest t such that $N_t(A) \geq N_t(B)$. Then $k < n$ and $N_t(A) < N_t(B)$ for $k + 1 \leq t \leq n$. Choose $x \in \{i \in B \backslash A \mid d_i = k + 1\}$.

$$\text{Then,} \quad N_t(A \cup \{x\}) = N_t(A) \leq t, \quad \text{for } 1 \leq t \leq k,$$

$$\text{and} \quad N_t(A \cup \{x\}) = N_t(A) + 1 \leq N_t(B) \leq t, \quad \text{for } k + 1 \leq t \leq n.$$

Thus $A \cup \{x\} \in \mathbf{C}$. □

The Algorithm

Implementing Greedy-MAX, for any given set of tasks S, we could sort them by penalties and determine the best selections.

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Time Complexity: $O(n^2)$.

Sort the tasks takes $O(n \log n)$.

Check whether $A \cup \{x\} \in \mathbf{C}$ takes $O(n)$.

There are totally $O(n)$ iterations of independence check.

Thus the finally complexity is $O(n \log n + n \cdot n) \rightarrow O(n^2)$.

An Example

Given an instance of 7 tasks with deadlines and penalties as follows:

a_i	1	2	3	4	5	6	7
d_i	4	2	4	3	1	4	6
w_i	70	60	50	40	30	20	10

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Greedy-MAX selects a_1, a_2, a_3, a_4, then rejects a_5, a_6, and finally accepts a_7.

The final schedule is $\langle a_2, a_4, a_1, a_3, a_7, a_5, a_6 \rangle$.

The optimal penalty is $w_5 + w_6 = 50$