Graph Decomposition*

Xiaofeng Gao

Department of Computer Science and Engineering
Shanghai Jiao Tong University, P.R.China

X033533-Algorithm: Analysis and Theory

*Special Thanks is given to Prof. Yijia Chen for sharing his teaching materials.

Outline

Exploring Graphs

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Algorithm 2: EXPLORE(G, v)

Input: $G = (V, E)$ is a graph; $v \in V$

Output: VISITED(u) is set to *true* for all nodes u **reachable** from v

VISITED(v) $=$ *true*;
PREVIST(v);
for *each edge* $(v, u) \in E$ **do**
    if *not* VISITED(u) **then**
        EXPLORE(G, u);
POSTVISIT(v);

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```

Note: PREVISIT and POSTVISIT procedures are optional. They work on a vertex when it is first discovered and left for the last time.

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Theorem: $\textsc{Explore}(G, v)$ is correct, i.e., it visits exactly all nodes that are reachable from v.

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If there is some u that EXPLORE misses, choose any path from v to u, and look at the last vertex v on that path that the procedure actually visited.

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So z was visited but w was not. This is a contradiction: while EXPLORE was at node z, it would have noticed w and moved on to it.

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Algorithm 5: DFS(G, v)

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Output: $\text{VISITED}(v)$ is set to *true* for all nodes $v \in V$

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VISITED(v) = true;
foreach v ∈ V do
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foreach v ∈ V do
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In step 2, over the course of the entire DFS, each edge $\{x, y\} \in E$ is examined exactly *twice*, once during EXPLORE(G, x) and once during EXPLORE(G, y). The overall time for step 2 is therefore $O(|E|)$.

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Thus the depth-first search has a running time of $O(|V| + |E|)$.

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Each time the DFS outer loop calls EXPLORE, a new connected component is picked out.

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where cc needs to be initialized to zero and to be incremented each time the DFS procedure calls EXPLORE.

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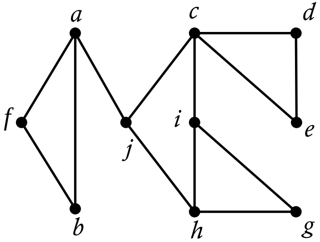
Lemma: For any nodes u and v, the two intervals $[\text{PRE}(u), \text{POST}(u)]$ and $[\text{PRE}(u), \text{POST}(u)]$ are either disjoint or one is contained within the other.

An executing example

Assume we use alphabetical order to explore G:



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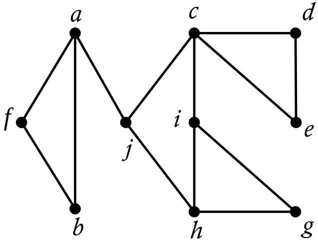

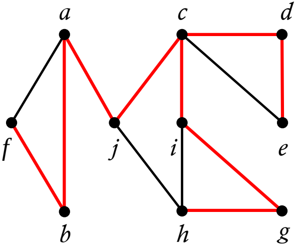

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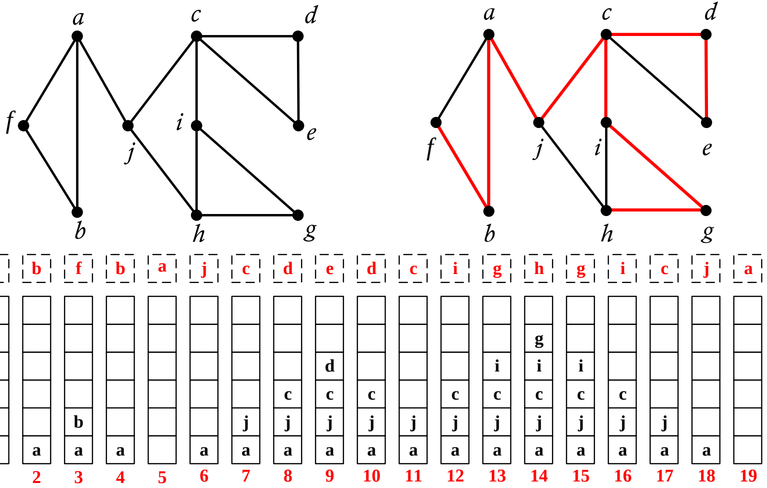

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- **Cross edges** lead to neither descendant nor ancestor; they therefore lead to a node that has already been completely explored (that is, already postvisited).

Types of edges (cont'd)

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PRE/POST ordering for (u, v)	Edge type
$[_u \quad [_v \quad\quad]_v \quad]_u$	Tree/forward
$[_v \quad [_u \quad\quad]_u \quad]_v$	Back
$[_v \quad]_v \quad\quad [_u \quad]_u$	Cross

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   - Directed Acyclic Graphs
   - Strongly Connected Components
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In particular, the edge $v_{i-1} \to v_i$ (or $v_k \to v_0$ if $i = 0$) is a back edge.

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The guaranteed existence of a source suggests an alternative approach to linearization:

1. Find a source, output it, and delete it from the graph.
2. Repeat until the graph is empty.

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(B) How do we continue once this first component has been discovered?

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Hence the strongly connected components can be linearized by arranging them in decreasing order of their highest POST numbers.

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So, if we do a depth-first search of G^R, the node with the highest POST number will come from a source strongly connected component in G^R, which is to say a sink strongly connected component in G.

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Therefore we can keep using the post numbering from our initial depth-first search on G^R to successively output the second strongly connected component, the third strongly connected component, and so on.

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2. Run the undirected connected components algorithm on G, and during the depth-first search, process the vertices in decreasing order of their POST numbers from step 1.

Outline

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Algorithm 7: $\textsc{BFS}(G, s)$

Input: Graph $G = (V, E)$, directed or undirected; vertex $s \in V$

Output: For all vertices u reachable from s, $\textsc{dist}(u)$ is set to the distance from s to u

foreach $u \in V$ **do**
  $\textsc{dist}(u) = \infty$;
$\textsc{dist}(s) = 0$; $Q = [s]$ (queue containing just s);
while *Q is not empty* **do**
  $u = \textsc{eject}(Q)$;
  foreach *edge* $(u, v) \in E$ **do**
    if $\textsc{dist}(v) = \infty$ **then**
      $\textsc{inject}(Q, v)$; $\textsc{dist}(v) = \textsc{dist}(u) + 1$;

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Lemma: For each $d = 0, 1, 2, \ldots$, there is a moment at which (1) all nodes at distance $\leq d$ from s have their distances correctly set; (2) all other nodes have their distances set to ∞; and (3) the queue contains exactly the nodes at distance d.

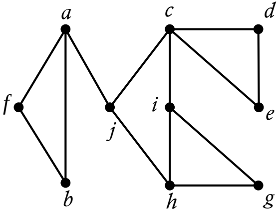
Correctness and efficiency

Lemma: For each $d = 0, 1, 2, \ldots$, there is a moment at which (1) all nodes at distance $\leq d$ from s have their distances correctly set; (2) all other nodes have their distances set to ∞; and (3) the queue contains exactly the nodes at distance d.

Lemma: BFS has a running time of $O(|V| + |E|)$.

An executing example

Assume we use alphabetical order to explore G:



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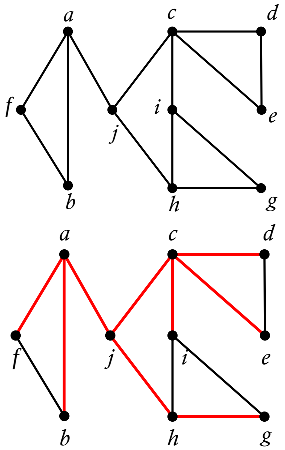

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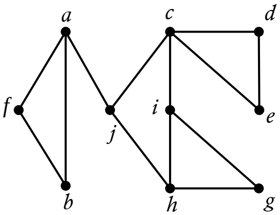

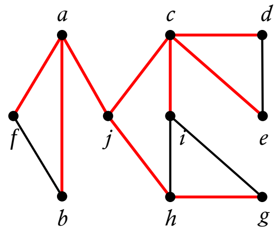

1		a					
2	a	b	f	j			
3	b	f	j				
4	f	j					
5	j	c	h				
6	c	h	d	e	i		
7	h	d	e	i	g		
8	d	e	i	g			
9	e	i	g				
10	i	g					
11	g						