Recursive and Recursively Enumerable Sets*

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Outline

Decision Problem, Predicate, Number Set

The following emphasizes the importance of the subsets of $\mathbb{N}$:

$$\begin{array}{rcl} \text{Decision Problems} & \Leftrightarrow & \text{Predicates on Number} \\ & \Leftrightarrow & \text{Sets of Numbers} \end{array}$$

A central theme of recursion theory is to look for sensible classification of number sets.

Classification is often done with the help of reduction.

Recursive Set

Let A be a subset of $\mathbb{N}$. The characteristic function of A is given by

$$c_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

A is recursive if $c_A(x)$ is computable.

Solvable Problem

A recursive set is (the domain of) a solvable problem.

It is important to know if a problem is solvable.

Examples

The following sets are recursive.

(a) $\mathbb{N}$.

(b) $\mathbb{E}$ (the even numbers).

(c) Any finite set.

(d) The set of prime numbers.

Unsolvable Problem

Here are some important unsolvable problems:

$$\begin{aligned}
K &= \{x \mid x \in W_x\}, \\
Fin &= \{x \mid W_x \text{ is finite}\}, \\
Inf &= \{x \mid W_x \text{ is infinite}\}, \\
Cof &= \{x \mid W_x \text{ is cofinite}\}, \\
Rec &= \{x \mid W_x \text{ is recursive}\}, \\
Tot &= \{x \mid \phi_x \text{ is total}\}, \\
Ext &= \{x \mid \phi_x \text{ is extensible to a total recursive function}\}.
\end{aligned}$$

Cofinite

$Cof = \{x \mid W_x \text{ is cofinite}\}$ means the set whose complement is finite.

Example 1: $\{x \mid x \geq 5\}$ is cofinite.

Not every infinite set is cofinite.

Example 2: $\mathbb{E}$, $\mathbb{O}$ are not cofinite.

Extensible Functions

$Ext = \{x \mid \phi_x$ is extensible to a total recursive function$\}$.

Example: $f(x) = \phi_x(x) + 1$ is not extensible.

Proof: Assume $f(x)$ is extensible, then define total recursive function

$$g(x) = \begin{cases} \psi_U(x,x) + 1 & \text{if } \psi_U(x,x) \text{ is defined.} \\ \maltese & \text{otherwise} \end{cases} \quad (1)$$

Let ϕ_m be the Gödel coding of $g(x)$, then ϕ_m is a total recursive function.

When $x = m$, $\phi_m(m) = \psi_U(m,m)$ by universal problem.

However, $\phi_m(m) = g(m) = \psi_U(m,m) + 1$ by equation (1). A contradiction. □

Comment: Not every partial recursive function can be obtained by restricting a total recursive function.

Decidable Predicate

A predicate $M(\mathbf{x})$ is decidable if its characteristic function $c_M(\mathbf{x})$ given by

$$c_M(\mathbf{x}) = \begin{cases} 1, & \text{if } M(\mathbf{x}) \text{ holds,} \\ 0, & \text{if } M(\mathbf{x}) \text{ does not hold.} \end{cases}$$

is computable.

The predicate $M(\mathbf{x})$ is undecidable if it is not decidable.

Recursive Set $\Leftrightarrow$ Solvable Problem $\Leftrightarrow$ Decidable Predicate.

Algebra of Decidability

Theorem. If A, B are recursive sets, then so are the sets $\overline{A}$, $A \cap B$, $A \cup B$, and $A \backslash B$.

Proof.

$c_{\overline{A}} = 1 \dot{-} c_A$.

$c_{A \cap B} = c_A \cdot c_B$.

$c_{A \cup B} = \max(c_A, c_B)$.

$c_{A \backslash B} = c_A \cdot c_{\overline{B}}$.

Reduction between Problems

A reduction is a way of defining a solution of a problem with the help of the solutions of another problem.

In recursion theory we are only interested in reductions that are computable.

There are several ways of reducing a problem to another.

The differences between different reductions from A to B consists in the manner and extent to which information about B is allowed to settle questions about A.

Many-One Reduction

The set A is many-one reducible, or m-reducible, to the set B if there is a total computable function f such that

$$x \in A \text{ iff } f(x) \in B$$

for all x.

We shall write $A \leq_m B$ or more explicitly $f : A \leq_m B$.

If f is injective, then it is a one-one reducibility, denoted by $\leq_1$.

Many-One Reduction

1. $\leq_m$ is reflexive and transitive.
2. $A \leq_m B$ iff $\overline{A} \leq_m \overline{B}$.
3. $A \leq_m \mathbb{N}$ iff $A = \mathbb{N}$; $A \leq_m \varnothing$ iff $A = \varnothing$.
4. $\mathbb{N} \leq_m A$ iff $A \neq \varnothing$; $\varnothing \leq_m A$ iff $A \neq \mathbb{N}$.

Non-Recursive Set

Proposition. $K = \{x \mid x \in W_x\}$ is not recursive.

Proof. If K were recursive, then the characteristic function

$$c(x) = \begin{cases} 1, & \text{if } x \in W_x, \\ 0, & \text{if } x \notin W_x, \end{cases}$$

would be computable.
Then the function $g(x)$ defined by

$$g(x) = \begin{cases} 0, & \text{if } c(x) = 0, \\ \text{undefined}, & \text{if } c(x) = 1. \end{cases}$$

would also be computable.
Let m be an index for g. Then

$$m \in W_m \text{ iff } c(m) = 0 \text{ iff } m \notin W_m.$$

Non-Recursive Set

Proposition. Neither $Tot = \{x \mid \phi_x \text{ is total}\}$ nor $\{x \mid \phi_x \simeq \mathbf{0}\}$ is recursive.

Proof. Consider the function f defined by

$$f(x,y) = \begin{cases} 0, & \text{if } x \in W_x, \\ \text{undefined}, & \text{if } x \notin W_x. \end{cases}$$

By S-m-n Theorem there is a primitive recursive function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

It is clear that $k : K \leq_m Tot$ and $k : K \leq_m \{x \mid \phi_x \simeq \mathbf{0}\}$.

Rice Theorem

Henry Rice.

Classes of Recursively Enumerable Sets and their Decision Problems. Transactions of the American mathematical Society, **77**:358-366, 1953.

Rice Theorem

Rice Theorem. (1953)

If $\emptyset \subsetneq \mathscr{B} \subsetneq \mathscr{C}_1$, then $\{x \mid \phi_x \in \mathscr{B}\}$ is not recursive.

Proof. Suppose $f_\emptyset \notin \mathscr{B}$ and $g \in \mathscr{B}$. Let the function f be defined by

$$f(x,y) = \begin{cases} g(y), & \text{if } x \in W_x, \\ \text{undefined}, & \text{if } x \notin W_x. \end{cases}$$

By S-m-n Theorem there is some primitive recursive function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

It is clear that k is a many-one reduction from K to $\{x \mid \phi_x \in \mathscr{B}\}$.

Applying Rice Theorem

According to Rice Theorem the following sets are non-recursive:

$$\begin{aligned} Fin &= \{x \mid W_x \text{ is finite}\}, \\ Inf &= \{x \mid W_x \text{ is infinite}\}, \\ Cof &= \{x \mid W_x \text{ is cofinite}\}, \\ Rec &= \{x \mid W_x \text{ is recursive}\}, \\ Tot &= \{x \mid \phi_x \text{ is total}\} \end{aligned}$$

Remark on Rice Theorem

Rice Theorem deals with programme independent properties.

It talks about classes of computable functions rather than classes of programmes.

All non-trivial semantic problems are algorithmically undecidable.

It is of no help to a proof that the set of all polynomial time Turing Machines is undecidable.

Recursively Enumerable Set

The partial characteristic function of a set A is given by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ \text{undefined}, & \text{if } x \notin A. \end{cases}$$

A is recursively enumerable if $\chi_A(x)$ is computable.

Notation 1: A is also called semi-recursive set, semi-computable set.

Notation 2: subsets of $\mathbb{N}^n$ can be defined as r.e. by coding to r.e. subsets of $\mathbb{N}$.

Partially Decidable Predicate

A predicate $M(\mathbf{x})$ of natural number is partially decidable if its partial characteristic function

$$\chi_M(\mathbf{x}) = \begin{cases} 1, & \text{if } M(\mathbf{x}) \text{ holds}, \\ \text{undefined}, & \text{if } M(\mathbf{x}) \text{ does not hold}, \end{cases}$$

is computable.

Partially Decidable Problem

A problem $f : \mathbb{N} \to \{0, 1\}$ is partially decidable if $dom(f)$ is r.e.

$$\begin{aligned} \text{Partially Decidable Problem} &\Leftrightarrow \text{Partially Decidable Predicate} \\ &\Leftrightarrow \text{Recursively Enumerable Set} \end{aligned}$$

Quick Review

Theorem. A predicate $M(\mathbf{x})$ is partially decidable iff there is a computable function $g(x)$ such that $M(\mathbf{x}) \Leftrightarrow \mathbf{x} \in Dom(g)$.

Theorem. A predicate $M(\mathbf{x})$ is partially decidable iff there is a decidable predicate $R(\mathbf{x}, y)$ such that $M(\mathbf{x}) \Leftrightarrow \exists y.R(\mathbf{x}, y)$.

Theorem. If $M(\mathbf{x}, y)$ is partially decidable, so is $\exists y.M(\mathbf{x}, y)$.

Corollary. If $M(\mathbf{x}, \mathbf{y})$ is partially decidable, so is $\exists \mathbf{y}.M(\mathbf{x}, \mathbf{y})$.

Theorem. $M(\mathbf{x})$ is decidable iff both $M(\mathbf{x})$ and $\neg M(\mathbf{x})$ are partially decidable.

Theorem. Let $f(\mathbf{x})$ be a partial function. Then f is computable iff the predicate '$f(\mathbf{x}) \simeq y$' is partially decidable.

Some Important Decidable Predicates

For each $n \geq 1$, the following predicates are primitive recursive:

1. $\mathsf{S}_n(e, \mathbf{x}, y, t) \stackrel{\text{def}}{=}$ '$P_e(\mathbf{x}) \downarrow y$ in t or fewer steps'.
2. $\mathsf{H}_n(e, \mathbf{x}, t) \stackrel{\text{def}}{=}$ '$P_e(\mathbf{x}) \downarrow$ in t or fewer steps'.

They are defined by

$$\begin{aligned} \mathsf{S}_n(e, \mathbf{x}, y, t) &\stackrel{\text{def}}{=} \mathsf{j}_n(e, \mathbf{x}, t) = 0 \wedge (\mathsf{c}_n(e, \mathbf{x}, t))_1 = y, \\ \mathsf{H}_n(e, \mathbf{x}, t) &\stackrel{\text{def}}{=} \mathsf{j}_n(e, \mathbf{x}, t) = 0. \end{aligned}$$

Example

1. The halting problem is partially decidable. Its partial characteristic function is given by

$$\chi_H(x, y) = \begin{cases} 1, & \text{if } P_x(y) \downarrow, \\ \text{undefined}, & \text{otherwise.} \end{cases}$$

2. $K = \{x \mid x \in W_x\}$ is r.e., but not recursive.
Proof: $\chi_K(x) = \mathbf{1}(\psi_U(x, x))$.

3. $\overline{K} = \{x \mid x \notin W_x\}$ is not r.e., (also not recursive).
Proof: If yes, then define $f(x) = \begin{cases} 1 & \text{if } x \notin W_x \\ \uparrow & \text{if } x \in W_x \end{cases}$
Then $x \in Dom(f) \Leftrightarrow x \notin W_x$. f is computable while $Dom(f)$ doesn't equal to any computable function. Contradiction!

Example (Cont.)

4. Any recursive set is r.e.

5. $\{x \mid W_x \neq \varnothing\}$ is r.e.
Proof: $W_x \neq \varnothing \Leftrightarrow \exists y \exists t (P_x(y) \downarrow$ in t steps$)$.

6. If f is a computable function, then $Ran(f)$ is r.e.
Proof: Let ϕ_m be the Gödel coding of f.

$$x \in E_m \Leftrightarrow \exists y \exists t (P_m(y) \downarrow x \text{ in } t \text{ steps}).$$

$$x \in E_m \text{ is partial decidable } \Leftrightarrow Ran(f) \text{ is r.e.}$$

Index Theorem

Theorem. A set is r.e. iff it is the domain of a unary computable function.

Proof:
"$\Rightarrow$": A is r.e. $\Rightarrow$ χ_A is computable $\Rightarrow$ "$x \in A \Leftrightarrow x \in \chi_A$".
Thus A is the domain of unary computable function χ_A.

"$\Leftarrow$": If f is a unary computable function, let $A = Dom(f)$.
Then $\chi_A = \mathbf{1}(f(x))$, which is computable.

Notation (Index for Recursively Enumerable Set): $W_0, W_1, W_2, \ldots$ is a repetitive enumeration of all r.e. sets. e is an index of A if $A = W_e$, end every r.e. set has an infinite number of indexes.

Normal Form Theorem

Theorem. The set A is r.e. iff there is a primitive recursive predicate $R(\mathbf{x}, y)$ such that $\mathbf{x} \in A$ iff $\exists y.R(\mathbf{x}, y)$.

Proof. "$\Leftarrow$": If $R(\mathbf{x}, y)$ is primitive recursive and $\mathbf{x} \in A \Leftrightarrow \exists y.R(\mathbf{x}, y)$, then define $g(\mathbf{x}) = \mu y R(\mathbf{x}, y)$.
Then $g(\mathbf{x})$ is computable and $\mathbf{x} \in A \Leftrightarrow \mathbf{x} \in Dom(g)$.

"$\Rightarrow$": suppose A is r.e., then χ_A is computable. Let P be program to compute χ_A and $R(\mathbf{x}, y)$ be

$$P(\mathbf{x}) \downarrow \text{ in } y \text{ steps.}$$

Then $R(\mathbf{x}, y)$ is primitive recursive (decidable) and $\mathbf{x} \in A \Leftrightarrow \exists y.R(\mathbf{x}, y)$.

Quantifier Contraction Theorem

Theorem (Applying the Normal Form Theorem). If $M(\mathbf{x}, \mathbf{y})$ is partially decidable, so is $\exists \mathbf{y}.M(\mathbf{x}, \mathbf{y})$ ($\{\mathbf{x} \mid \exists \mathbf{y}.M(\mathbf{x}, \mathbf{y})\}$ is r.e.).

Proof. Let $R(\mathbf{x}, \mathbf{y}, z)$ be a primitive recursive predicate such that

$$M(\mathbf{x}, \mathbf{y}) \Leftrightarrow \exists z.R(\mathbf{x}, \mathbf{y}, z).$$

Then $\exists \mathbf{y}.M(\mathbf{x}, \mathbf{y}) \Leftrightarrow \exists \mathbf{y}.\exists z.R(\mathbf{x}, \mathbf{y}, z) \Leftrightarrow \exists u.R(\mathbf{x}, (u)_0, \cdots, (u)_{m+1})$.
$(u = 2^{y_1}3^{y_2}\cdots p_m^{y_m}, p_{m+1}^z$, if $\mathbf{y} = (y_1, \cdots, y_m))$.

By Normal Form Theorem, $\exists \mathbf{y}.M(\mathbf{x}, \mathbf{y})$ is partially decidable, and $\{\mathbf{x} \mid \exists \mathbf{y}.M(\mathbf{x}, \mathbf{y})\}$ is r.e.

Uniformisation Theorem

Theorem (Applying the Normal Form Theorem). If $R(x, y)$ is partially decidable, then there is a computable function $c(x)$ such that $c(x)\downarrow$ iff $\exists y.R(x, y)$ and $c(x)\downarrow$ implies $R(x, c(x))$.

We may think of $c(x)$ as a choice function for $R(x, y)$. The theorem states that the choice function is computable.

Complementation Theorem

Theorem. A is recursive iff A and $\overline{A}$ are r.e.

Proof. "$\Rightarrow$": If A is recursive, then χ_A and $\chi_{\overline{A}}$ are computable.
Thus $\Rightarrow$ A and $\overline{A}$ are r.e.

"$\Leftarrow$": Suppose A and $\overline{A}$ are r.e. Then some primitive recursive predicates $R(x,y), S(x,y)$ exist such that

$$\begin{array}{rcl} x \in A & \Leftrightarrow & \exists y R(x,y), \\ x \in \overline{A} & \Leftrightarrow & \exists y S(x,y). \end{array}$$

Now let $f(x) = \mu y(R(x,y) \vee S(x,y))$.

Since either $x \in A$ or $x \in \overline{A}$ holds, $f(x)$ is total and computable, and $x \in A \Leftrightarrow R(x, f(x))$. Thus $x \in A$ is decidable $\Rightarrow$ A is recursive.

The Hardest Recursively Enumerable Set

Fact. If $A \leq_m B$ and B is r.e. then A is r.e..

Theorem. A is r.e. iff $A \leq_m K$.

Proof. Suppose A is r.e. Let $f(x,y)$ be defined by

$$f(x,y) = \begin{cases} 1, & \text{if } x \in A, \\ \text{undefined}, & \text{if } x \notin A. \end{cases}$$

By S-m-n Theorem there is a total computable function $s(x)$ such that $f(x,y) = \phi_{s(x)}(y)$. It is clear that $x \in A$ iff $s(x) \in K$.

No r.e. set is more difficult than K.

Applying Complementation Theorem

Proposition. If A is r.e. but not recursive, then $\overline{A} \not\leq_m A \not\leq_m \overline{A}$.

It contradicts to our intuition that A and $\overline{A}$ are equally difficult.

Graph Theorem

Theorem. Let $f(x)$ be a partial function. Then $f(x)$ is computable iff the predicate '$f(x) \simeq y$' is partially decidable iff $\{\pi(x,y) \mid f(x) \simeq y\}$ is r.e.

Proof. If $f(x)$ is computable by $P(x)$, then

$$f(x) \simeq y \Leftrightarrow \exists t.(P(x) \downarrow y \text{ in } t \text{ steps}).$$

The predicate '$P(x) \downarrow y$ in t steps' is primitive recursive.

Conversely let $R(x,y,t)$ be such that

$$f(x) \simeq y \Leftrightarrow \exists t.R(x,y,t).$$

Now $f(x) = \mu y.R(x,y,\mu t.R(x,y,t))$.

Listing Theorem

Listing Theorem. A is r.e. iff either $A = \varnothing$ or A is the range of a unary total computable function.

Proof. Suppose A is nonempty and its partial characteristic function is computed by P. Let a be a member of A. The total function $g(x,t)$ given by

$$g(x,t) = \begin{cases} x, & \text{if } P(x)\downarrow \text{ in } t \text{ steps}, \\ a, & \text{if otherwise.} \end{cases}$$

is computable. Clearly A is the range of $h(z) = g((z)_1,(z)_2)$.

The converse follows from Graph Theorem.
Suppose $A = Ran(h)$, then

$$x \in A \Leftrightarrow \exists \mathbf{y}(h(\mathbf{y}) \simeq x) \Leftrightarrow \exists \mathbf{y} \exists t (P(\mathbf{y}) \downarrow x \text{ in } t \text{ steps})$$

Listing Theorem

It gives rise to the terminology recursively enumerable.

The elements of a r.e. set can be effectively generated. E.g., A can be enumerated as $A = \{h(0), h(1), \cdots, h(n), \cdots\}$, where h is a primitive recursive function.

$\{E_0, E_1, \cdots, E_n, \cdots\}$ is another enumeration of all r.e. sets.

R.e. set are effectively generated sets, which is a list compiled by an informal effective procedure (may go on ad infinitum).

An Example

The set $\{x \mid \text{if there is a run of exactly } x \text{ consecutive 7's in the decimal expansion of } \pi\}$ is r.e.

Proof. Run an algorithm that computes successive digits in the decimal expansion of π. Each time a run of 7s appears, count the number of consecutive 7s in the run and add this number to the list.

Applying Listing Theorem

A set is r.e. iff it is the range of a computable function.

Equivalence Theorem. Let $A \subseteq \mathbb{N}$. Then the following are equivalent:
(a). A is r.e.
(b). $A = \varnothing$ or A is the range of a unary total computable function.
(c). A is the range of a (partial) computable function.

Applying Listing Theorem

Theorem. Every infinite r.e. set has an infinite recursive subset.

Proof. Suppose $A = Ran(f)$ where f is a total computable function. An infinite recursive subset is enumerated by the total increasing computable function g given by

$$\begin{aligned} g(0) &= f(0), \\ g(n+1) &= f(\mu y(f(y) > g(n))). \end{aligned}$$

(g is total since $A = Ran(f)$ is infinite. g is computable by minimalisation and recursion).
$Ran(g)$ is an infinite recursive subset of A.

Applying Listing Theorem

Theorem. An infinite set is recursive iff it is the range of a total increasing computable function (if it can be recursively enumerated in increasing order).

Proof."$\Rightarrow$" Suppose A is recursive and infinite. Then A is enumerated by the increasing function f given by

$$\begin{aligned} f(0) &= \mu y(y \in A), \\ f(n+1) &= \mu y(y \in A \wedge y > f(n)). \end{aligned}$$

f is total since A is infinite. f is computable by minimalisation and recursion. $Ran(g)$ is an infinite recursive subset of A.

"$\Leftarrow$": Suppose A is the range of the computable total increasing function f; i.e., $f(0) < f(1) < f(2) < \cdots$ It is clear that if $y = f(n)$ then $n \leq y$. Hence

$$y \in A \Leftrightarrow y \in Ran(f) \Leftrightarrow \exists n \leq y (f(n) = y)$$

Applying Listing Theorem

Theorem. The set $\{x \mid \phi_x \text{ is total}\}$ is not r.e.

Proof. If $\{x \mid \phi_x \text{ is total}\}$ were a r.e. set, then there would be a total computable function f whose range is the r.e. set.

The function $g(x)$ given by $g(x) = \phi_{f(x)}(x) + 1$ would be total and computable.

An Alternative Proof

Let $f(x,y) =$

$$\begin{cases} 1 & \text{if } P_x(x) \text{ does not converge in } y \text{ or fewer steps,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Since $f(x,y)$ is computable by Church's Thesis, from s-m-n theorem, there is a total computable function $k(x)$, such that $\phi_{k(x)}(y) \simeq f(x,y)$.

From the definition of f, we have

$$\begin{cases} x \in W_x \Rightarrow (\exists y)(P_x(x) \text{ converges in } y \text{ steps}) \Rightarrow \phi_{k(x)} \text{ is not total} \\ x \notin W_x \Rightarrow (\forall y)(P_x(x) \text{ does not converge in } y \text{ steps}) \Rightarrow \phi_{k(x)} \text{ is total} \end{cases}$$

Therefore, '$x \notin W_x$' iff. '$\phi_{k(x)}$ is total'. We have 'ϕ_x is total' is not partially computable.

Closure Theorem

Theorem. The recursively enumerable sets are closed under union and intersection uniformly and effectively.

Proof. According to S-m-n Theorem there are primitive recursive functions $r(x,y), s(x,y)$ such that

$$\begin{aligned} W_{r(x,y)} &= W_x \cup W_y, \\ W_{s(x,y)} &= W_x \cap W_y. \end{aligned}$$

Rice-Shapiro Theorem

Rice-Shapiro Theorem. Suppose that $\mathscr{A}$ is a set of unary computable functions such that the set $\{x \mid \phi_x \in \mathscr{A}\}$ is r.e. Then for any unary computable function f, $f \in \mathscr{A}$ iff there is a finite function $\theta \subseteq f$ with $\theta \in \mathscr{A}$.

Proof of Rice-Shapiro Theorem

Suppose $A = \{x \mid \phi_x \in \mathscr{A}\}$ is r.e.

Suppose $f \in \mathscr{A}$ but $\forall$ finite $\theta \subseteq f. \theta \notin \mathscr{A}$.

Let P be a partial characteristic function of K.
Define the computable function $g(z,t)$ by

$$g(z,t) \simeq \begin{cases} f(t), & \text{if } P(z) \not\downarrow \text{ in } t \text{ steps}, \\ \uparrow, & \text{otherwise}. \end{cases}$$

According to S-m-n Theorem, there is a primitive recursive function $s(z)$ such that $g(z,t) \simeq \phi_{s(z)}(t)$.

By construction $\phi_{s(z)} \subseteq f$ for all z.

$z \in K \Rightarrow \phi_{s(z)}$ is finite $\Rightarrow s(z) \notin A$;
$z \notin K \Rightarrow \phi_{s(z)} = f \Rightarrow s(z) \in A$.

Proof of Rice-Shapiro Theorem

Suppose f is a computable function and there is a finite $\theta \in \mathscr{A}$ such that $\theta \subseteq f$ and $f \notin \mathscr{A}$.

Define the computable function $g(z,t)$ by

$$g(z,t) \simeq \begin{cases} f(t), & \text{if } t \in Dom(\theta) \vee z \in K, \\ \uparrow, & \text{otherwise}. \end{cases}$$

According to S-m-n Theorem, there is a primitive recursive function $s(z)$ such that $g(z,t) \simeq \phi_{s(z)}(t)$.

$z \in K \Rightarrow \phi_{s(z)} = f \Rightarrow s(z) \notin A$;
$z \notin K \Rightarrow \phi_{s(z)} = \theta \Rightarrow s(z) \in A$.

Reversing Rice-Shapiro Theorem

$\{x \mid \phi_x \in \mathscr{A}\}$ is r.e. if the following hold:

(1) $\Theta = \{g(\theta) \mid \theta \in \mathscr{A}$ and θ is finite$\}$ is r.e., where g is a canonical encoding of the finite functions.

(2) $\forall f \in \mathscr{A}$, $\exists$ finite $\theta \in \mathscr{A}$, $\theta \subseteq f$.

Corollary

The sets $\{x \mid \phi_x$ is total$\}$ and $\{x \mid \phi_x$ is not total$\}$ are not r.e.

Proof. Consider the set $\mathscr{A} = \{f \mid f \in \mathscr{C}_1 \wedge f$ is total$\}$. For no $f \in \mathscr{A}$ is there a finite $\theta \subseteq f$ with $\theta \in \mathscr{A}$. Hence $\{x \mid \phi_x$ is total$\}$ is not r.e.

Consider the set $\mathscr{B} = \{f \mid f \in \mathscr{C}_1 \wedge f$ is not total$\}$. Then if f is any total computable function, $f \notin \mathscr{B}$; but every finite function $\theta \subseteq f$ is in $\mathscr{B}$. Hence $\{x \mid \phi_x$ is not total$\}$ is not r.e. by Rice-Shapiro theorem.

Applying Rice-Shapiro Theorem

The following sets are not recursively enumerable:

$$\begin{aligned}
Fin &= \{x \mid W_x \text{ is finite}\},\\
Inf &= \{x \mid W_x \text{ is infinite}\},\\
Cof &= \{x \mid W_x \text{ is cofinite}\},\\
Rec &= \{x \mid W_x \text{ is recursive}\},\\
Tot &= \{x \mid \phi_x \text{ is total}\},\\
Con &= \{x \mid \phi_x \text{ is total and constant}\},\\
Ext &= \{x \mid \phi_x \text{ is extensible to a total recursive function}\}.
\end{aligned}$$

Non-r.e. Sets

Target. We consider non-r.e. sets to form *creative sets*. Suppose A is any non-r.e. set, then if W_x is an r.e. set contained in A, there must be a number $y \in A \backslash W_x$. This number y is a witness of $A \neq W_x$.

Example. Consider $\overline{K} = \{x \mid x \notin W_x\}$

Suppose $W_x \subseteq \overline{K}$. Then $x \in \overline{K} \setminus W_x$. So x is a witness that the inclusion $W_x \subseteq \overline{K}$ is strict.

We call $\overline{K}$ productive.

Productive Sets

Definition. A set A is productive if there is a total computable function g such that whenever $W_x \subseteq A$, then $g(x) \in A \setminus W_x$.

The function is called a productive function for A.

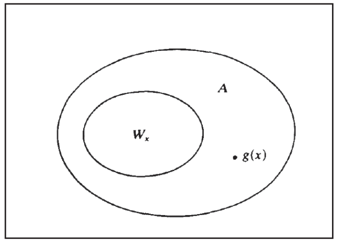


Fig. A productive set

Notation. A productive set is not r.e.

Example. $\overline{K}$ is productive with productive function $g(x) = x$.

Reduction Theorem

Theorem. Suppose that A and B are sets such that A is productive, and there is a total computable function such that $x \in A$ iff $f(x) \in B$. Then B is productive.

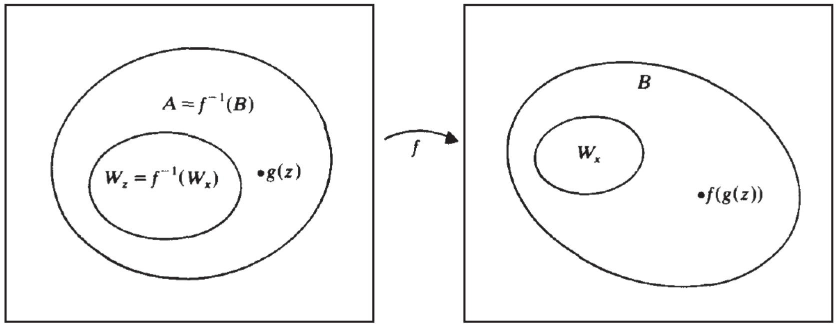
Proof. Suppose $W_x \subseteq B$. Then $W_z = f^{-1}(W_x) \subseteq f^{-1}(B) = A$ for some z.

Moreover, $f^{-1}(W_x)$ is r.e. (by substitution), so there is a z such that $f^{-1}(W_x) = W_z$. Now $W_z \subseteq A$, and $g(z) \in A \setminus W_z$. Hence $f(g(z)) \in B \setminus W_x$.

$f(g(z))$ is a witness to the fact that $W_x \neq B$.

We now need to obtain the witness $f(g(z))$ effectively from x. Apply the s-m-n theorem to $\phi_x(f(y))$, one gets a total computable function $k(x)$ such that $\phi_{k(x)}(y) = \phi_x(f(y))$. Then $W_{k(x)} = f^{-1}(W_x)$. It follows that $f(g(k(x))) \in B \setminus W_x$.

Proof



Examples

1. $\{x \mid \phi_x \neq \mathbf{0}\}$ is productive.

Proof. $f(x,y) = \begin{cases} 0 & \text{if } x \in W_x \\ \uparrow & \text{if } x \notin W_x \end{cases}$. Reduce from $\overline{K}$.

2. $\{x \mid c \notin W_x\}$ is productive.

Proof. $f(x,y) = \begin{cases} y & \text{if } x \in W_x \\ \uparrow & \text{if } x \notin W_x \end{cases}$. Reduce from $\overline{K}$.

3. $\{x \mid c \notin E_x\}$ is productive.

Application of Rich's Theorem

Theorem. Suppose that $\mathscr{B}$ is a set of unary computable functions with $f_\varnothing \in \mathscr{B}$ and $\mathscr{B} \neq \mathscr{C}_1$. Then the set $B = \{x \mid \phi_x \in \mathscr{B}\}$ is productive.

Proof. Choose a computable function $g \notin \mathscr{B}$. Consider function f defined by

$$f(x,y) = \begin{cases} g(y), & \text{if } x \in W_x, \\ \uparrow, & \text{if } x \notin W_x. \end{cases}$$

By s-m-n theorem there is some total computable function $k(x)$ such that $\phi_{k(x)}(y) \simeq f(x,y)$.

It is clear that $x \in W_x$ iff $\phi_{k(x)} = g$ iff $\phi_{k(x)} \notin \mathscr{B}$. Thus $x \in \overline{K}$ iff $k(x) \in B$.

Example. $\{x \mid \phi_x \text{ is not total}\}$ is productive.
($\mathscr{B} = \{f \mid f \in \mathscr{C}_1 \wedge f \text{ is not total}\}$.)

Creative Sets

Definition. A set A is creative if it is r.e. and its complement $\overline{A}$ is productive.

Example. K is creative. (The simplest example of a creative set).

Notation. From the theorem that A is recursive $\Leftrightarrow$ A and $\overline{A}$ are r.e. we can say that a creative set is an r.e. set that fails to be recursive in a very strong way. (Creative sets are r.e. sets having the most difficult decision problem.)

Examples

1. $\{x \mid c \in W_x\}$ is creative.

2. $\{x \mid c \in E_x\}$ is creative.

3. $A = \{x \mid \phi_x(x) = 0\}$ is creative.

Proof. A is r.e.

To obtain a productive function for $\overline{A}$, by s-m-n theorem one gets a total computable function $g(x)$ such that $\phi_{g(x)}(y) = 0 \Leftrightarrow \phi_x(y)$ is defined.

Then $g(x) \in A \Leftrightarrow g(x) \in W_x$. So if $W_x \subseteq \overline{A}$ we must have $g(x) \in \overline{A} \setminus W_x$.

Thus g is a productive function for $\overline{A}$.

Application of Rice's Theorem

Theorem. Suppose that $\mathscr{A} \subseteq \mathscr{C}_1$ and let $A = \{x \mid \phi_x \in \mathscr{A}\}$. If A is r.e. and $A \neq \varnothing, \mathbb{N}$, then A is creative.

Proof. Suppose A is r.e. and $A \neq \varnothing,\ \mathbb{N}$.

If $f_\varnothing \in \mathscr{A}$, then A is productive by the previous theorem. This is a contradiction.

Thus $f_\varnothing \notin \mathscr{A}$. $\overline{A}$ is productive by the same theorem. Hence A is creative.

Examples

1. $A = \{x \mid c \in W_x\}$ is creative. It corresponds to $\mathscr{A} = \{f \in \mathscr{C}_1 \mid f(c)\downarrow\}$.

2. $A = \{x \mid c \in E_x\}$ is creative. It corresponds to $\mathscr{A} = \{f \in \mathscr{C}_1 \mid \exists x(f(x)\downarrow c)\}$.

3. $A = \{x \mid W_x \neq \varnothing\}$ is creative. It corresponds to $\mathscr{A} = \{f \in \mathscr{C}_1 \mid f \neq f_\varnothing\}$.

Discussion

Question. Are all non-recursive r.e. sets creative?

The answer is negative. By a special construction we can obtain r.e.sets that are neither recursive nor creative.

Subset Theorem

Lemma. Suppose that g is a total computable function. Then there is a total computable function k such that for all x, $W_{k(x)} = W_x \cup \{g(x)\}$.

Proof. Using the s-m-n theorem, take $k(x)$ to be a total computable function such that

$$\phi_{k(x)}(y) = \begin{cases} 1, \text{if } y \in W_x \vee y = g(x), \\ \uparrow, \text{otherwise} \end{cases}.$$

Subset Theorem

Theorem. A productive set contains an infinite r.e. subset.

Proof. Let A be a productive set with productive function g. The idea is to enumerate a non-repetitive infinite set $B = \{y_0, y_1, \cdots\} \subseteq A$.

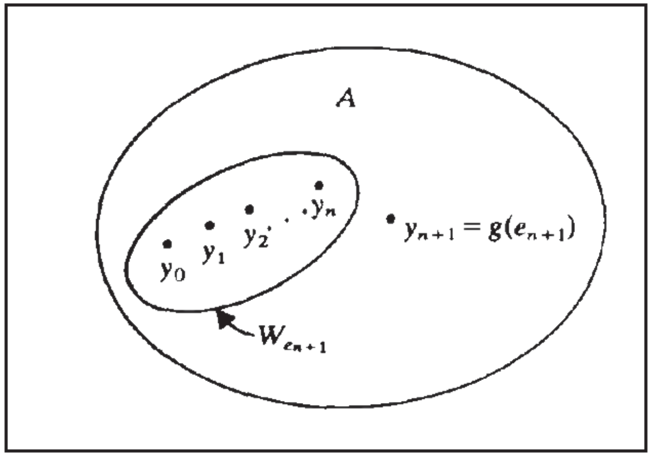
Take e_0 to be some index for $W_{e_0} = \varnothing$. Since $W_{e_0} \subseteq A$, $g(e_0) \in A$. Put $y_0 = g(e_0) \in A$.

For $n \geq 0$, assume $\{y_0, \cdots, y_n\} \subseteq A$. Find an e_{n+1} s.t. $\{y_0, \cdots, y_n\} = W_{e_{n+1}} \subseteq A$. Then $g(e_{n+1}) \in A \backslash W_{e_{n+1}}$. Thus if we put $y_{n+1} = g(e_{n+1})$, we have $y_{n+1} \in A$ and $y_{n+1} \neq y_0, \cdots, y_n$.

By the Lemma there is some total computable function k such that for all x, $W_{k(x)} = W_x \cup \{g(x)\}$. So the infinite set $\{e_0, \ldots, k^n(e_0), \ldots\}$ is r.e.

It follows that the infinite set $\{g(e_0), \ldots, g(k^n(e_0)), \ldots\}$ is a r.e. subset of A.

Illumination



Corollary

If A is creative, then $\overline{A}$ contains an infinite r.e. subset.

Simple Sets

Definition. A set A is simple if

(i) A is r.e.,

(ii) $\overline{A}$ is infinite,

(iii) $\overline{A}$ contains no infinite r.e. subset.

Simple Sets

Theorem. A simple set is neither recursive nor creative.

Proof. Since $\overline{A}$ can not be r.e., A can not be recursive.

(iii) implies that A can not be creative.

Simple Sets

Theorem. There is a simple set.

Proof. Define $f(x) = \phi_x(\mu z(\phi_x(z) > 2x))$. Let A be $Ran(f)$.

(i) A is r.e.

(ii) $\overline{A}$ is infinite. This is because $A \cap \{0, 1, \ldots, 2n\}$ contains at most the elements $\{f(0), f(1), \ldots, f(n-1)\}$.

(iii) Suppose B is an infinite r.e. set. Then there is a total computable function ϕ_b such that $B = E_b$. Since ϕ_b is total, $f(b)$ is defined and $f(b) \in A$. Hence $B \not\subseteq \overline{A}$.