Reducibility and Degree*

Xiaofeng Gao

Department of Computer Science and Engineering
Shanghai Jiao Tong University, P.R.China

CS363-Computability Theory

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The differences between different reductions consists in the manner and extent to which information about B is allowed to settle questions about A.

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If f is injective, then we are talking about one-one reducibility.

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$$\phi_{k(x)} = \mathbf{0} \circ \phi_x$$

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3. $A \leq_m B$ iff $\overline{A} \leq_m \overline{B}$.

   If $g : A \leq_m B$, then $x \in A \Leftrightarrow f(x) \in B$; so $x \in \overline{A} \Leftrightarrow g(x) \in \overline{B}$.
   Hence $g : \overline{A} \leq_m \overline{B}$.

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   Let $b \in B$, $c \notin B$, $f(x) = \begin{cases} b & \text{if } x \in A; \\ c & \text{if } x \notin A. \end{cases}$; then f is computable.

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   Let $g : B \leq_m A$, $A = Dom(h)$, $(h \in \mathscr{C}_1)$; then $B = Dom(h \circ g)$ (B is r.e.)

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   (i). “$\Rightarrow$": Let $f : \mathbb{N} \leq_m A$, then $A = Ran(f)$, so $A \neq \varnothing$ (f is total).
   (i). “$\Leftarrow$": If $A \neq \varnothing$, choose $c \in A$. If $g(x) = c$, we have $g : \mathbb{N} \leq_m A$.
   (ii). $\varnothing \leq_m A \Leftrightarrow \mathbb{N} \leq_m \overline{A} \Leftrightarrow \overline{A} \neq \varnothing \Leftrightarrow A = \mathbb{N}$.

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Proof. By contradiction, if $\{x \mid \phi_x \text{ is total}\} \leq_m K$, and K is r.e., then $\{x \mid \phi_x \text{ is total}\}$ is r.e. (same as $\{x \mid \phi_x \text{ is not total}\}$).

However, by Rice-Shapiro Theorem, Neither $\{x \mid \phi_x \text{ is total}\}$ nor $\{x \mid \phi_x \text{ is not total}\}$ is r.e.

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Notation: It contradicts to our intuition that A and $\overline{A}$ are equally difficult.

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Proof. "$\Leftarrow$". Since $A \leq_m K$, and K is r.e., then A is r.e.

Suppose A is r.e. Let $f(x,y)$ be $f(x,y) = \begin{cases} 1, & \text{if } x \in A, \\ \uparrow, & \text{if } x \notin A. \end{cases}$

By s-m-n Theorem $\exists s(x) : \mathbb{N} \to \mathbb{N}$ such that $f(x,y) = \phi_{s(x)}(y)$.

It is clear that $x \in A$ iff $\phi_{s(x)}(s(x))$ is defined iff $s(x) \in K$. Hence $A \leq_m K$.

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Notation. K is the most difficult partially decidable problem.

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Definition. Two sets A, B are many-one equivalent, notation $A \equiv_m B$ (abbreviated m-equivalent), if $A \leq_m B$ and $B \leq_m A$.

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Theorem. $\equiv_m$ is an equivalence relation.

Proof.
(1). Reflexivity: $A \leq_m A \Rightarrow A \equiv_m A$.
(2). Symmetry: $A \equiv_m B \Rightarrow B \leq_m A,\ A \leq_m B \Rightarrow B \equiv_m A$.
(3). Transitivity: $A \equiv_m B,\ B \equiv_m C \Rightarrow A \leq_m C,\ C \leq_m A \Rightarrow A \equiv_m C$.

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"$\Leftarrow$": $f_{\mathbb{N}}(x,y) = \begin{cases} y & \text{if } x \in W_x \\ \uparrow & \text{if } x \notin W_x \end{cases} \Rightarrow K \leq_m \{x \mid c \in W_x\}$

"$\Rightarrow$": $\{x \mid c \in W_x\}$ is r.e., so $\{x \mid c \in W_x\} \leq_m K$.

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   $A \neq \varnothing, \mathbb{N} \Rightarrow \overline{A} \neq \varnothing, \mathbb{N}$.

   A is recursive, by previous theorem $A \leq_m \overline{A}$. Similarly, $\overline{A} \leq_m A$.

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   "$\Leftarrow$": $\phi_{k(x)} = \mathbf{0} \circ \phi_x \Rightarrow \{x \mid \phi_x \text{ is total}\} \leq_m \{x \mid \phi_x = \mathbf{0}\}$.

   "$\Rightarrow$": Let $\phi_{k(x)}(y) = \begin{cases} 0 & \text{if } \phi_x(y) = 0; \\ \uparrow & \text{if } \phi_x(y) \neq 0. \end{cases}$.

   Then $\phi_x = \mathbf{0} \Leftrightarrow \phi_{k(x)}$ is total $\Rightarrow \{x \mid \phi_x = \mathbf{0}\} \leq_m \{x \mid \phi_x \text{ is total}\}$.

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A recursive m-degree is an m-degree that contains a recursive set.
An r.e. m-degree is an m-degree that contains an r.e. set.

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Definition (Partial Order on m-Degree). Let $\mathbf{a}$, $\mathbf{b}$ be m-degrees.

(1). $\mathbf{a} \leq_m \mathbf{b}$ iff $A \leq_m B$ for some $A \in \mathbf{a}$ and $B \in \mathbf{b}$.

(2). $\mathbf{a} <_m \mathbf{b}$ iff $\mathbf{a} \leq_m \mathbf{b}$ and $\mathbf{b} \not\leq_m \mathbf{a}$ ($\mathbf{a} \neq \mathbf{b}$).

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Notation. From the definition of $\equiv_m$,
$\mathbf{a} \leq_m \mathbf{b} \Leftrightarrow \forall A \in \mathbf{a}, B \in \mathbf{b}, A \leq_m B$.

Theorem

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Proof.

(1) By transitivity $\mathbf{a} \leq_m \mathbf{b}$, $\mathbf{b} \leq_m \mathbf{c}$ implies $\mathbf{a} \leq_m \mathbf{c}$.

If $\mathbf{a} \leq_m \mathbf{b}$ and $\mathbf{b} \leq_m \mathbf{a}$, we have to prove that $\mathbf{a} = \mathbf{b}$.

(2) Irreflexivity: Let $A \in \mathbf{a}$ and $B \in \mathbf{b}$, then we have $A \leq_m B$ and $B \leq_m A$, so $A \equiv_m B$. Hence $\mathbf{a} = \mathbf{b}$.

Consequently, $<_m$ is partial ordering.

Some Facts

1. **o** and **n** are respectively the recursive m-degrees $\{\emptyset\}$ and $\{\mathbb{N}\}$.
   $A \leq_m \mathbf{N} \Leftrightarrow A = \mathbf{N}$; $A \leq_m \emptyset \Leftrightarrow A = \emptyset$.

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2. The recursive m-degree $\mathbf{0}_m$ consists of all the recursive sets except $\varnothing, \mathbb{N}$.

   $\mathbf{0}_m \leq_m \mathbf{a}$ for any m-degree **a** other than **o** and **n**.

   A is recursive, $B \leq_m A \Rightarrow B$ is recursive;

   A is recursive and $B \neq \varnothing, \mathbb{N} \Rightarrow A \leq_m B$.

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1. **o** and **n** are respectively the recursive m-degrees $\{\emptyset\}$ and $\{\mathbb{N}\}$.

   $A \leq_m \mathbf{N} \Leftrightarrow A = \mathbf{N}$; $A \leq_m \emptyset \Leftrightarrow A = \emptyset$.

2. The recursive m-degree $\mathbf{0}_m$ consists of all the recursive sets except $\emptyset, \mathbb{N}$.

   $\mathbf{0}_m \leq_m \mathbf{a}$ for any m-degree $\mathbf{a}$ other than **o** and **n**.

   A is recursive, $B \leq_m A \Rightarrow B$ is recursive;

   A is recursive and $B \neq \emptyset, \mathbb{N} \Rightarrow A \leq_m B$.

3. $\forall$ m-degree **a**, $\mathbf{o} \leq_m \mathbf{a}$ provided $\mathbf{a} \neq \mathbf{n}$; $\mathbf{n} \leq_m \mathbf{a}$ provided $\mathbf{a} \neq \mathbf{o}$.

   $\mathbb{N} \leq_m A \Leftrightarrow A \neq \emptyset$; $\emptyset \leq_m A \Leftrightarrow A \neq \mathbb{N}$.

Facts (2)

4. An r.e. m-degree consists of only r.e. sets.

   If A is r.e. and $B \leq_m A$, then B is r.e.

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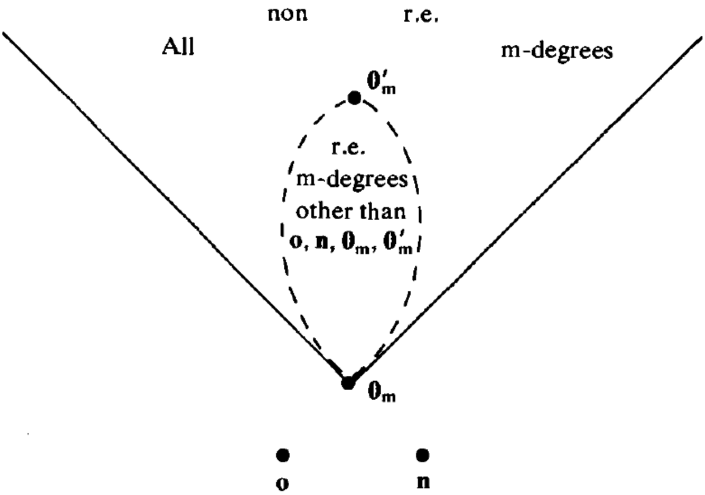
   If A is r.e. and $B \leq_m A$, then B is r.e.

6. The maximum r.e. m-degree $d_m(K)$ is denoted by $\mathbf{0}'_m$.

   A set A is r.e. iff $A \leq_m K$.

Illumination



Facts about r.e. *m*-Degrees

1. Excluding **o** and **n**, there is a minimum r.e. m-degree $\mathbf{0}_m$ (in fact $\mathbf{0}_m$ is minimum among all m-degrees).

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3. There is a maximum r.e. m-degree $\mathbf{0}'_m$.
4. While there are uncountably many m-degrees, only countably many of these are r.e.

Algebraic Structure

Theorem. The *m*-degrees form an upper semi-lattice.

Group

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closure: $a, b \in G \Rightarrow a \bullet b \in G$.

associativity: $(a \bullet b) \bullet c = a \bullet (b \bullet c)$.

identity: $\forall a \in G$, $\exists$ identity element $e \in G$, s.t. $e \bullet a = a \bullet e = a$.

invertibility: $\forall a \in G$, $\exists$inverse $b \in G$ s.t. $a \bullet b = b \bullet a = e$ $(b = a^{-1})$.

Lattice

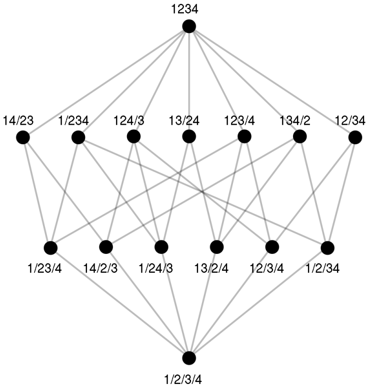
In mathematics, a lattice is a partially ordered set (poset) $(L, \leq)$ in which any two elements have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet).

To qualify as a lattice, the set and the operation must satisfy tow conditions: join-semilattice, meet-semilattice.

join-semilattice:	$\forall a, b \in L$, the set $\{a, b\}$ has a join $a \vee b$. (the least upper bound)
meet-semilattice:	$\forall a, b \in L$, the set $\{a, b\}$ has a meet $a \wedge b$. (the greatest lower bound)

The Name “Lattice"

The name "lattice" is suggested by the form of the Hasse diagram depicting it. I.e., the right picture is the lattice of partitions of a four-element set $\{1, 2, 3, 4\}$, ordered by the relation "is a refinement of".



Upper Semi-lattice

Theorem. Any pair of m-degrees **a**, **b** have a least upper bound; i.e. there is an m-degree **c** such that

(i). $\mathbf{a} \leq_m \mathbf{c}$ and $\mathbf{b} \leq_m \mathbf{c}$ (**c** is an upper bound);

(ii). $\mathbf{c} \leq_m$ any other upper bound of **a**, **b**.

Proof

(i). Pick $A \in \mathbf{a}$, $B \in \mathbf{b}$, and let $C = A \oplus B$, i.e.,

$$C = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\}.$$

Then

$$x \in A \Leftrightarrow 2x \in C \Longrightarrow A \leq_m C;$$

$$x \in B \Leftrightarrow 2x+1 \in C \Longrightarrow B \leq_m C;$$

Thus $\mathbf{c}$ is an upper bound of $\mathbf{a}$, $\mathbf{b}$.

Proof (2)

(ii). Let $\mathbf{d}$ is an m-degree such that $\mathbf{a} \leq_m \mathbf{d}$, and $\mathbf{b} \leq_m \mathbf{d}$.

$\forall D \in \mathbf{d}$, suppose $f : A \leq_m D$ and $g : B \leq_m D$. Then

$$\begin{aligned} x \in C &\Leftrightarrow (x \text{ is even } \& \frac{x}{2} \in A) \vee (x \text{ is odd } \& \frac{x-1}{2} \in B) \\ &\Leftrightarrow (x \text{ is even } \& f(\frac{x}{2}) \in D) \vee (x \text{ is odd } \& g(\frac{x-1}{2}) \in D) \end{aligned}$$

Thus we have $h : C \leq_m D$ if we define $h = \begin{cases} f(\frac{x}{2}) & \text{if } x \text{ is even;} \\ g(\frac{x-1}{2}) & \text{if } x \text{ is odd.} \end{cases}$

Hence $\mathbf{c} \leq_m \mathbf{d}$.

Definition

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Notation. $\mathbf{0}'_m$, the m-degree of K is maximum among all r.e. m-degrees, and thus K is m-complete r.e. set (or just called m-complete set).

Theorem

Theorem. The following statements are valid.

(i) K is m-complete.

(ii) A is m-complete iff $A \equiv_m K$ iff A is r.e. and $K \leq_m A$.

(iii) $\mathbf{0}'_m$ consists exactly of all the m-complete sets.

Examples

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(iv). $\{x \mid x \in E_x\}$.

Creative Set

Theorem. Any m-complete set is creative.

Creative Set

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Proof. If A is m-complete, A is r.e. set.

Also, $K \leq_m A$, so $\overline{K} \leq_m \overline{A}$. Thus $\overline{A}$ is productive.

Myhill's Theorem

Myhill's Theorem. A set is m-complete iff it is creative.

m-Complete r.e. Sets

Corollary. If $\mathbf{a}$ is the m-degree of any simple set, then $\mathbf{0}_m <_m \mathbf{a} <_m \mathbf{0}'_m$ (Simple sets are not m-complete).

m-Complete r.e. Sets

Corollary. If $\mathbf{a}$ is the m-degree of any simple set, then $\mathbf{0}_m <_m \mathbf{a} <_m \mathbf{0}'_m$ (Simple sets are not m-complete).

Proof. Simple sets are designed to be neither recursive nor creative.

Outline

Comparison

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(i) The exceptional behavior of $\varnothing$ and $\mathbb{N}$.

(ii) The invalidity of $A \not\equiv_m \overline{A}$ in general.

The problem is due to the restricted use of oracles.

E.g. $x \in \overline{A}$ iff $x \notin A$

Relative Computability

Suppose χ is a total unary function.

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Such an algorithm is called a χ-algorithm.

URMO - Unlimited Register Machine with Oracle

A URM with oracle, URMO for short, can recognize a fifth kind of instruction, $O(n)$, for every $n \geq 1$.

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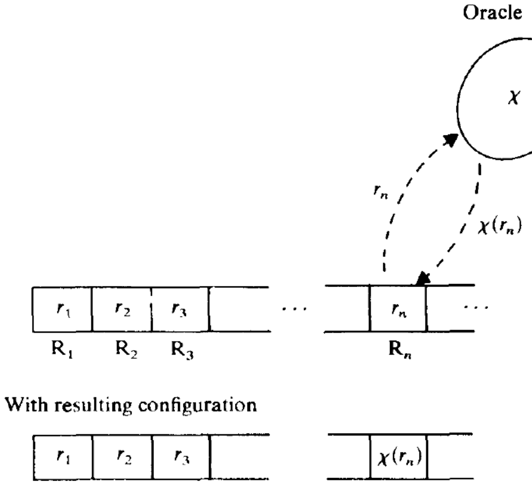
$P^{\chi}(\mathbf{a}) \downarrow b$ means the computation $P^{\chi}(\mathbf{a})$ with initial configuration $a_1, a_2, \cdots, a_n, 0, 0, \cdots$ stops with the number b is register R_1.

Illumination



URMO-Computable

Let χ be a unary total function, and suppose the f is a partial function from $\mathbb{N}^n$ to $\mathbb{N}$.

(a) Let P be a URMO program, then P URMO-computes f relative to χ (or f is χ-computed by P) if, for every $\mathbf{a} \in \mathbb{N}^n$ and $b \in \mathbb{N}$, $P^\chi(\mathbf{a}) \downarrow b$ iff $f(\mathbf{a}) \simeq b$.

(b) The function f is URMO-computable relative to χ (or χ-computable) if there is a URMO program that URMO-computes it relative to χ.

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Facts

(i) $\chi \in \mathscr{C}^{\chi}$.

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Since $\mathscr{C} \subseteq \mathscr{C}^{\chi}$, we need to prove $\mathscr{C}^{\chi} \subseteq \mathscr{C}$. χ is computable, then whenever a value of χ is requested simply compute it by the algorithm for χ. By Church's thesis, f is computable.

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Construct corresponding URMO programs.

(v) If ψ is a total unary function that is χ-computable, then $\mathscr{C}^{\psi} \subseteq \mathscr{C}^{\chi}$.

By Church's thesis.

Partial Recursive Function

The class $\mathscr{R}^\chi$ of χ-partial recursive functions is the smallest class of functions such that

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χ-recursive, χ-primitive recursive are defined in the obvious way.

Theorem. For any χ, $\mathscr{R}^\chi = \mathscr{C}^\chi$.

Numbering URMO programs

Let's fix an effective enumeration of all URMO programs

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W_m^{χ} is $Dom(\phi_m^{\chi})$ and E_m^{χ} is $Ran(\phi_m^{\chi})$.

Numbering URMO programs

S-m-n Theorem. For each $m, n \geq 1$ there is a total computable $(m+1)$-ary function $s_n^m(e, \mathbf{x})$ such that for any χ

$$\phi_e^{\chi, m+n}(\mathbf{x}, \mathbf{y}) \simeq \phi_{s_n^m(e,\mathbf{x})}^{\chi, n}(\mathbf{y}).$$

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Notice that $s_n^m(e, \mathbf{x})$ does not refer to χ.

Universal Programs for Relative Computability

Universal Function Theorem. For each n, the universal function $\psi_U^{\chi,n}$ for n-ary χ-computable functions given by

$$\psi_U^{\chi,n}(e,\mathbf{x}) \simeq \phi_e^{\chi,n}(\mathbf{x})$$

is χ-computable.

Relativization

Once we have the S-m-n Theorem and the Universal Function Theorem, we can do the recursion theory relative to an oracle.

χ-Recursive and χ-r.e. Sets

Let A be a set

(a) A is χ-recursive if c_A is χ-computable.

(b) A is χ-r.e. if the partial characteristic function
$$f(x) = \begin{cases} 1 & \text{if } x \in A, \\ \uparrow & \text{if } x \notin A \end{cases}$$ is χ-computable.

χ-Recursive and χ-r.e. Sets

Theorem. The following statements are valid.

(i) For any set A, A is χ-recursive iff A and $\overline{A}$ are χ-r.e.

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(i) For any set A, A is χ-recursive iff A and $\overline{A}$ are χ-r.e.

(ii) For any set A, the following are equivalent.

- A is χ-r.e.
- $A = W_m^\chi$ for some m.
- $A = E_m^\chi$ for some m.
- $A = \emptyset$ or A is the range of a total χ-computable function.
- For some χ-decidable predicate $R(x, y)$, $x \in A$ iff $\exists y.R(x, y)$.

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- For some χ-decidable predicate $R(x, y)$, $x \in A$ iff $\exists y.R(x, y)$.

(iii) $K^\chi \overset{\text{def}}{=} \{x \mid x \in W_x^\chi\}$ is χ-r.e. but not χ-recursive.

Computability Relative to a Set

Computability relative to a set A means computability relative to its characteristic function c_A.

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For example:

P^A for P^{c_A} (if P is a URMO program),

$\mathscr{C}^A$ for $\mathscr{C}^{c_A}$,

ϕ_m^A for $\phi_m^{c_A}$.

W_m^A for $W_m^{c_A}$,

E_m^A for $E_m^{c_A}$,

K^A for K^{c_A},

A-recursive for c_A-recursive

A-r.e. for c_A-r.e.

...

Outline

Turing Reducibility and Turing Degrees

The set A is Turing reducible to B, notation $A \leq_T B$, if A has a B-computable characteristic function c_A.

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The set A is Turing reducible to B, notation $A \leq_T B$, if A has a B-computable characteristic function c_A.

The sets A, B are Turing equivalent, notation $A \equiv_T B$, if $A \leq_T B$ and $B \leq_T A$.

Notation

Suppose $A \leq_T B$ and P is the URMO program that computes c_A relative to B. Then $\forall x$, $P^B(x)$ converges and

$$P^B(x) \downarrow 1 \text{ if } x \in A$$
$$P^B(x) \downarrow 0 \text{ if } x \notin A$$

When calculating $P^B(x)$ there will be a finite number of requests to the oracle for a value $c_B(n)$ of c_B. These requests amount to a finite number of questions of the form '$n \in B$?'.

So for any x, '$x \in A$?' is settled in a mechanical way by answering a finite number of questions about B.

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$c_{\overline{A}} = \overline{sg} \circ c_A$, $\overline{A}$ is A-recursive $\Longrightarrow \overline{A} \leq_T A$. (Similarly $A \leq_T \overline{A}$.)

Facts. (2)

(v) If A is recursive, then $A \leq_T B$ for all B.

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(v) If A is recursive, then $A \leq_T B$ for all B.

Since $\mathscr{C} \subseteq \mathscr{C}^\chi$.

(vi) If B is recursive and $A \leq_T B$, then A is recursive.

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If $A \leq_m B$ then $A \leq_T B$; A set A is r.e. iff $A \leq_m K$.

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The relation $\leq$ is a partial order.

Theorem

(i) There is precisely one recursive degree $\mathbf{0}$, which consists of all the recursive sets and is the unique minimal degree.

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(ii) Let $\mathbf{0}'$ be the degree of K. Then $\mathbf{0} < \mathbf{0}'$ and $\mathbf{0}'$ is a maximum among all r.e. degrees.

From (i), $\mathbf{0} \leq \mathbf{0}'$; $\mathbf{0} \neq \mathbf{0}'$ since K is not recursive. Since A is r.e. $\Rightarrow$ $A \leq_T K$, we have if $\mathbf{a}$ is any r.e. degree, $\mathbf{a} \leq \mathbf{0}'$.

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(iii) $d_m(A) \subseteq d_T(A)$; and if $d_m(A) \leq_m d_m(B)$ then $d_T(A) \leq d_T(B)$.

If $A \leq_m B$ then $A \leq_T B$.

Jump Operation

Theorem. The following statements are valid.

(i) $K^A \stackrel{\text{def}}{=} \{x \mid x \in W_x^A\}$ is A-r.e.

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(iii) If A is recursive then $K^A \equiv_T K$.

"$\Leftarrow$" $K \leq_T K^A$ since K is A-r.e. for any A;

"$\Rightarrow$" If A is recursive then A-computable partial characteristic function of K^A is actually computable (if χ is computable, then $\mathscr{C} = \mathscr{C}^\chi$). Hence K^A is r.e., and $K^A \leq_T K$.

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(iv) $A <_T K^A$.

"$A \leq_T K^A$" is given by (ii). "$A \not\equiv_T K^A$" is given by "K^χ is χ-r.e. but not χ-recursive."

Relativization

(v) If $A \leq_T B$ then $K^A \leq_T K^B$.

If $A \leq_T B$, then since K^A is A-r.e. it is also B-r.e., so $K^A \leq_T K^B$.

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(vi) If $A \equiv_T B$ then $K^A \equiv_T K^B$.

Follows immediately from (v).

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Notation (1). By Relativization jump is a valid definition because the degree of K^A is the same for every $A \in \mathbf{a}$.

Notation (2). The new definition of $\mathbf{0}'$ as the jump of $\mathbf{0}$ accords with our earlier definition of $0'$ as the degree of K.

Basic Properties

Theorem. For any degree $\mathbf{a}$ and $\mathbf{b}$, the following statements are valid.

(i) $\mathbf{a} < \mathbf{a}'$.

(ii) If $\mathbf{a} < \mathbf{b}$ then $\mathbf{a}' < \mathbf{b}'$

(iii) If $B \in \mathbf{b}$, $A \in \mathbf{a}$ and B is A-r.e. then $\mathbf{b} \leq \mathbf{a}'$.

Important Results

Theorem. Any degrees $\mathbf{a}, \mathbf{b}$ have a unique least upper bound.

Theorem. Any non-recursive r.e. degree contains a simple set.

Theorem. There are r.e. sets A, B s.t. $A \not\leq_T B$ and $B \not\leq_T A$. Hence, if $\mathbf{a}$, $\mathbf{b}$ are $d_T(A)$, $d_T(B)$ respectively, $\mathbf{a} \not\leq \mathbf{b}$ and $\mathbf{b} \not\leq \mathbf{a}$, and thus $\mathbf{0} < \mathbf{a} < \mathbf{0}'$ and $\mathbf{0} < \mathbf{b} < \mathbf{0}'$.

Degrees $\mathbf{a}$, $\mathbf{b}$ such that $\mathbf{a} \not\leq \mathbf{b}$ and $\mathbf{b} \not\leq \mathbf{a}$ are called incomparable degrees, denoted as $\mathbf{a} \mid \mathbf{b}$.

Theorem. For any r.e. degree $\mathbf{a} > \mathbf{0}$, there is an r.e. degree $\mathbf{b}$ such that $\mathbf{b} \mid \mathbf{a}$.

Important Results (2)

Sack's Density Theorem. For any r.e. degrees $\mathbf{a} < \mathbf{b}$ there is an r.e. degree $\mathbf{c}$ with $\mathbf{a} < \mathbf{c} < \mathbf{b}$.

Sack's Splitting Theorem. For any r.e. degrees $\mathbf{a} > \mathbf{0}$ there are r.e. degrees $\mathbf{b}$, $\mathbf{c}$ such that $\mathbf{b} < \mathbf{a}$ $\mathbf{c} < \mathbf{a}$ and $\mathbf{a} = \mathbf{b} \cup \mathbf{c}$ (hence $\mathbf{b} \mid \mathbf{c}$).

Lachlan, Yates Theorem.
(a). $\exists$ r.e. degrees $\mathbf{a}$, $\mathbf{b} > \mathbf{0}$ such that $\mathbf{0}$ is the greatest lower bound of $\mathbf{a}$ and $\mathbf{b}$.
(b). $\exists$ r.e. degrees $\mathbf{a}$, $\mathbf{b}$ having no greatest lower bound (either among all degrees or among r.e. degrees).

Shoenfield Theorem. There is a non-r.e. degree $\mathbf{a} < \mathbf{0}'$.

Spector Theorem. There is a minimal degree. (A minimal degree is a degree $\mathbf{m} > \mathbf{0}$ such that there is no degree $\mathbf{a}$ with $\mathbf{0} < \mathbf{a} < \mathbf{m}$).

Theorem. For any r.e. m-degree $\mathbf{a} >_m \mathbf{0_m}$, $\exists$ an r.e. m-degree $\mathbf{b}$ s.t. $\mathbf{b} \mid \mathbf{a}$.