

Mathematical Logic (IV)

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1 The Semantics of First-order Logic

1.1 Isomorphisms

Definition 1.1. Let \mathfrak{A} and \mathfrak{B} be two S -structures.

(a) A mapping $\pi : A \rightarrow B$ is an **isomorphism from \mathfrak{A} to \mathfrak{B}** (in short $\pi : \mathfrak{A} \cong \mathfrak{B}$) if the following conditions are satisfied.

(i) π is a bijection.

(ii) For any n -ary relation symbol $R \in S$ and $a_0, \dots, a_{n-1} \in A$

$$(a_0, \dots, a_{n-1}) \in R^{\mathfrak{A}} \iff (\pi(a_0), \dots, \pi(a_{n-1})) \in R^{\mathfrak{B}}.$$

(iii) For any n -ary function symbol $f \in S$ and $a_0, \dots, a_{n-1} \in A$

$$\pi(f^{\mathfrak{A}}(a_0, \dots, a_{n-1})) = f^{\mathfrak{B}}(\pi(a_0), \dots, \pi(a_{n-1})).$$

(iv) For any constant $c \in S$

$$\pi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

(b) \mathfrak{A} and \mathfrak{B} are isomorphic, written $\mathfrak{A} \cong \mathfrak{B}$, if there is an isomorphism $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$. ←

Observe that the above definition is not symmetric. However we can easily show:

Lemma 1.2. \cong is an equivalence relation. That is, for all S -structures $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$

1. $\mathfrak{A} \cong \mathfrak{A}$;

2. $\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{B} \cong \mathfrak{A}$;

3. if $\mathfrak{A} \cong \mathfrak{B}$ and $\mathfrak{B} \cong \mathfrak{C}$, then $\mathfrak{A} \cong \mathfrak{C}$. ←

Lemma 1.3 (The Isomorphism Lemma). Let \mathfrak{A} and \mathfrak{B} be two isomorphic S -structures. Then for every S -sentence φ

$$\mathfrak{A} \models \varphi \iff \mathfrak{B} \models \varphi.$$

←

Proof: Let β be an assignment in \mathfrak{A} . By the coincidence lemma, it suffices to show that there is an assignment β' in \mathfrak{B} such that

$$(\mathfrak{A}, \beta) \models \varphi \iff (\mathfrak{B}, \beta') \models \varphi, \tag{1}$$

where φ is an S -sentence.

Let $\pi : \mathfrak{A} \cong \mathfrak{B}$ and we define an assignment β^π in \mathfrak{B} by

$$\beta^\pi(x) := \pi(\beta(x))$$

for any variable x . Then we prove for any **S-formula** φ

$$(\mathfrak{A}, \beta) \models \varphi \iff (\mathfrak{B}, \beta^\pi) \models \varphi, \quad (2)$$

which certainly generalizes (1). To simplify notation, let $\mathfrak{J} := (\mathfrak{A}, \beta)$ and $\mathfrak{J}^\pi := (\mathfrak{B}, \beta^\pi)$. First, it is routine to verify that for every S-term t

$$\pi(\mathfrak{J}(t)) = \mathfrak{J}^\pi(t). \quad (3)$$

Then we prove (2) by induction on the construction of S-formula φ .

- $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned} \mathfrak{J} \models t_1 \equiv t_2 &\iff \mathfrak{J}(t_1) = \mathfrak{J}(t_2) \\ &\iff \pi(\mathfrak{J}(t_1)) = \pi(\mathfrak{J}(t_2)) && \text{(since } \pi \text{ is an injection)} \\ &\iff \mathfrak{J}^\pi(t_1) = \mathfrak{J}^\pi(t_2) && \text{(by (3))} \\ &\iff \mathfrak{J}^\pi \models t_1 \equiv t_2. \end{aligned}$$

- $\varphi = R t_1 \cdots t_n$.

$$\begin{aligned} \mathfrak{J} \models R t_1 \cdots t_n &\iff (\mathfrak{J}(t_1), \dots, \mathfrak{J}(t_n)) \in R^{\mathfrak{A}} \\ &\iff (\pi(\mathfrak{J}(t_1)), \dots, \pi(\mathfrak{J}(t_n))) \in R^{\mathfrak{B}} \\ &\iff (\mathfrak{J}^\pi(t_1), \dots, \mathfrak{J}^\pi(t_n)) \in R^{\mathfrak{B}} && \text{(by (3))} \\ &\iff \mathfrak{J}^\pi \models R t_1 \cdots t_n. \end{aligned}$$

- $\varphi = \neg\psi$. It follows that $\mathfrak{J} \models \neg\psi \iff \mathfrak{J} \not\models \psi \iff \mathfrak{J}^\pi \not\models \psi \iff \mathfrak{J}^\pi \models \neg\psi$.
- $\varphi = \psi \vee \chi$. The inductive argument is similar to the above $\neg\psi$.
- $\varphi = \exists x\psi$. This is again the most complicated case.

$$\begin{aligned} \mathfrak{J} \models \exists x\psi &\iff \text{there exists an } a \in A \text{ such that } \mathfrak{J} \frac{a}{x} = \left(\mathfrak{A}, \beta \frac{a}{x} \right) \models \psi \\ &\iff \text{there exists an } a \in A \text{ such that } \left(\mathfrak{J} \frac{a}{x} \right)^\pi = \left(\mathfrak{A}, \beta \frac{a}{x} \right)^\pi \models \psi, \\ &\quad \left(\text{by induction hypothesis on } \mathfrak{J} \frac{a}{x}, \left(\mathfrak{J} \frac{a}{x} \right)^\pi, \text{ and } \psi \right) \\ &\quad \text{that is, there exists an } a \in A \text{ such that } \left(\mathfrak{B}, \beta^\pi \frac{\pi(a)}{x} \right) \models \psi \\ &\iff \text{there exists a } b \in B \text{ such that } \left(\mathfrak{B}, \beta^\pi \frac{b}{x} \right) \models \psi \quad \text{(since } \pi \text{ is surjective)} \\ &\quad \text{i.e., there exists a } b \in B \text{ with } \mathfrak{J}^\pi \frac{b}{x} = \left(\mathfrak{B}, \beta^\pi \right) \frac{b}{x} \models \psi \\ &\iff \mathfrak{J}^\pi \models \exists x\psi. \end{aligned}$$

This finishes the proof. □

Corollary 1.4. *Let $\pi : \mathfrak{A} \cong \mathfrak{B}$ and $\varphi \in L_n^S$. Then for every a_0, \dots, a_{n-1}*

$$\mathfrak{A} \models \varphi[a_0, \dots, a_{n-1}] \iff \mathfrak{B} \models \varphi[\pi(a_0), \dots, \pi(a_{n-1})]$$

□

1.2 Substitution

In mathematics, when writing $f(y + 10)$ we plug the value of $y + 10$ into $f(x)$. We will do the same for $\varphi(x)$ where we want to substitute x by a term t . This is not completely trivial, e.g.,

$$\varphi(x) = \exists z z + z \equiv x \quad \text{and} \quad t = x + z.$$

It is obviously wrong for

$$\exists z z + z \equiv x + z.$$

Definition 1.5. Let t be an S-term, x_0, \dots, x_r variables, and t_0, \dots, t_r S-terms. Then the term

$$t \frac{t_0, \dots, t_r}{x_0, \dots, x_r}$$

is defined inductively as follows.

(a) Let $t = x$ be a variable. Then

$$t \frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \begin{cases} t_i & \text{if } x = x_i \text{ for some } 0 \leq i \leq r \\ x & \text{otherwise.} \end{cases}$$

(b) For a constant $t = c$

$$c \frac{t_0, \dots, t_r}{x_0, \dots, x_r} := c.$$

(c) For a function term

$$ft'_1 \dots t'_n \frac{t_0, \dots, t_r}{x_0, \dots, x_r} := ft'_1 \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \dots t'_n \frac{t_0, \dots, t_r}{x_0, \dots, x_r}. \quad \dashv$$

Definition 1.6. Let φ be an S-formula, x_0, \dots, x_r variables, and t_0, \dots, t_r S-terms. We define

$$\varphi \frac{t_0, \dots, t_r}{x_0, \dots, x_r}$$

inductively as follow.

(a) Assume $\varphi = t'_1 \equiv t'_2$. Then

$$\varphi \frac{t_0, \dots, t_r}{x_0, \dots, x_r} := t'_1 \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \equiv t'_2 \frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(b) Let $\varphi = Rt'_1 \dots t'_n$. We set

$$\varphi \frac{t_0, \dots, t_r}{x_0, \dots, x_r} := Rt'_1 \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \dots t'_n \frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(c) For $\varphi = \neg\psi$

$$\varphi \frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \neg\psi \frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(d) For $\varphi = (\psi_1 \vee \psi_2)$

$$\varphi \frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \left(\psi_1 \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \vee \psi_2 \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \right).$$

- (e) Assume $\varphi = \exists x\psi$. Let x_{i_1}, \dots, x_{i_s} ($i_1 < \dots < i_s$) be the variables x_i in x_0, \dots, x_r with $x_i \in \text{free}(\exists x\psi)$ and $x_i \neq t_i$. In particular, $x \neq x_{i_1}, \dots, x \neq x_{i_s}$. Then

$$\varphi \frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \exists u \left[\psi \frac{t_{i_1}, \dots, t_{i_s}, u}{x_{i_1}, \dots, x_{i_s}, x} \right],$$

where $u = x$ if x does not occur in t_{i_1}, \dots, t_{i_s} ; otherwise u is the first variable in $\{v_0, v_1, v_2, \dots\}$ which does not occur in $\psi, t_{i_1}, \dots, t_{i_s}$. \dashv

Examples 1.7. 1.

$$[Pv_0fv_1v_2] \frac{v_2, v_0, v_1}{v_1, v_2, v_3} = Pv_0fv_2v_0.$$

2.

$$[\exists v_0 Pv_0fv_1v_2] \frac{v_4, fv_1v_1}{v_0, v_2} = \exists v_0 \left[Pv_0fv_1v_2 \frac{fv_1v_1, v_0}{v_2, v_0} \right] = \exists v_0 Pv_0fv_1fv_1v_1.$$

3.

$$[\exists v_0 Pv_0fv_1v_2] \frac{v_0, v_2, v_4}{v_1, v_2, v_0} = \exists v_3 \left[Pv_0fv_1v_2 \frac{v_0, v_3}{v_1, v_0} \right] = \exists v_3 Pv_3fv_0v_2. \quad \dashv$$

Definition 1.8. Let β be an assignment in \mathfrak{A} and $a_0, \dots, a_r \in A$. Then

$$\beta \frac{a_0, \dots, a_r}{x_0, \dots, x_r}$$

is an assignment in \mathfrak{A} defined by

$$\beta \frac{a_0, \dots, a_r}{x_0, \dots, x_r}(y) := \begin{cases} a_i & \text{if } y = x_i \text{ for } 0 \leq i \leq r \\ \beta(y) & \text{otherwise.} \end{cases}$$

For an S-interpretation $\mathfrak{J} = (\mathfrak{A}, \beta)$ we let

$$\mathfrak{J} \frac{a_0, \dots, a_r}{x_0, \dots, x_r} := \left(\mathfrak{A}, \beta \frac{a_0, \dots, a_r}{x_0, \dots, x_r} \right). \quad \dashv$$

Lemma 1.9 (The Substitution Lemma). (a) For every S-term t

$$\mathfrak{J} \left(\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \right) = \mathfrak{J} \frac{\mathfrak{J}(t_0), \dots, \mathfrak{J}(t_r)}{x_0, \dots, x_r}(t).$$

(b) For every S-formula φ

$$\mathfrak{J} \models \varphi \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \iff \mathfrak{J} \frac{\mathfrak{J}(t_0), \dots, \mathfrak{J}(t_r)}{x_0, \dots, x_r} \models \varphi. \quad \dashv$$

Proof: (a) Assume $t = x$. If $x \neq x_i$ for all $0 \leq i \leq r$, then

$$\frac{t_0, \dots, t_r}{x_0, \dots, x_r} = x.$$

Therefore,

$$\mathfrak{J} \left(\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \right) = \mathfrak{J}(x) = \mathfrak{J} \frac{\mathfrak{J}(t_0), \dots, \mathfrak{J}(t_r)}{x_0, \dots, x_r}(x) = \mathfrak{J} \frac{\mathfrak{J}(t_0), \dots, \mathfrak{J}(t_r)}{x_0, \dots, x_r}(t).$$

Otherwise, $x = x_i$ for some $0 \leq i \leq r$. Then $t_{x_0, \dots, x_r}^{t_0, \dots, t_r} = t_i$. It follows that

$$\mathcal{J} \left(t_{x_0, \dots, x_r}^{t_0, \dots, t_r} \right) = \mathcal{J}(t_i) = \mathcal{J} \frac{\mathcal{J}(t_0), \dots, \mathcal{J}(t_r)}{x_0, \dots, x_r} (x_i) = \mathcal{J} \frac{\mathcal{J}(t_0), \dots, \mathcal{J}(t_r)}{x_0, \dots, x_r} (t).$$

The other cases of t can be shown similarly.

(b) Assume that $\varphi = R t'_1 \dots t'_n$. Then

$$\begin{aligned} \mathcal{J} \models \varphi \frac{t_0, \dots, t_r}{x_0, \dots, x_r} &\iff \left(\mathcal{J} \left(t'_1 \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \right), \dots, \mathcal{J} \left(t'_n \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \right) \right) \in \mathbb{R}^{\mathfrak{A}} \\ &\iff \left(\mathcal{J} \frac{\mathcal{J}(t_0), \dots, \mathcal{J}(t_r)}{x_0, \dots, x_r} (t'_1), \dots, \mathcal{J} \frac{\mathcal{J}(t_0), \dots, \mathcal{J}(t_r)}{x_0, \dots, x_r} (t'_n) \right) \in \mathbb{R}^{\mathfrak{A}} \quad (\text{by (a)}) \\ &\iff \mathcal{J} \frac{\mathcal{J}(t_0), \dots, \mathcal{J}(t_r)}{x_0, \dots, x_r} \models R t'_1 \dots t'_n \\ &\quad \text{i.e., } \mathcal{J} \frac{\mathcal{J}(t_0), \dots, \mathcal{J}(t_r)}{x_0, \dots, x_r} \models \varphi. \end{aligned}$$

For another case, let $\varphi = \exists x \psi$. Again, let x_{i_1}, \dots, x_{i_s} be the variables x_i with $x_i \in \text{free}(\exists x \psi)$ and $x_i \neq t_i$. Choose u according to Definition 1.6 (e). In particular, u does not occur in t_{i_1}, \dots, t_{i_s} . Then

$$\begin{aligned} \mathcal{J} \models \varphi \frac{t_0, \dots, t_r}{x_0, \dots, x_r} &\iff \mathcal{J} \models \exists u \left[\psi \frac{t_{i_1}, \dots, t_{i_s}, u}{x_{i_1}, \dots, x_{i_s}, x} \right] \\ &\iff \text{there exists an } a \in A \text{ such that } \mathcal{J} \frac{a}{u} \models \psi \frac{t_{i_1}, \dots, t_{i_s}, u}{x_{i_1}, \dots, x_{i_s}, x} \\ &\iff \text{there exists an } a \in A \text{ such that } \left[\mathcal{J} \frac{a}{u} \right] \frac{\mathcal{J} \frac{a}{u} (t_{i_1}), \dots, \mathcal{J} \frac{a}{u} (t_{i_s}), \mathcal{J} \frac{a}{u} (u)}{x_{i_1}, \dots, x_{i_s}, x} \models \psi \\ &\quad (\text{by induction hypothesis}) \\ &\iff \text{there exists an } a \in A \text{ such that } \left[\mathcal{J} \frac{a}{u} \right] \frac{\mathcal{J}(t_{i_1}), \dots, \mathcal{J}(t_{i_s}), a}{x_{i_1}, \dots, x_{i_s}, x} \models \psi \\ &\quad (\text{by the coincidence lemma and that } u \text{ does not occur in } t_{i_1}, \dots, t_{i_s}) \\ &\iff \text{there exists an } a \in A \text{ such that } \mathcal{J} \frac{\mathcal{J}(t_{i_1}), \dots, \mathcal{J}(t_{i_s}), a}{x_{i_1}, \dots, x_{i_s}, x} \models \psi \\ &\quad (\text{by (either } u = x \text{ or } u \text{ does not occur in } \psi) \text{ and the coincidence lemma}) \\ &\iff \text{there exists an } a \in A \text{ such that } \left[\mathcal{J} \frac{\mathcal{J}(t_{i_1}), \dots, \mathcal{J}(t_{i_s})}{x_{i_1}, \dots, x_{i_s}} \right] \frac{a}{x} \models \psi \\ &\quad (\text{since } x \neq x_{i_1}, \dots, x \neq x_{i_s}) \\ &\iff \mathcal{J} \frac{\mathcal{J}(t_{i_1}), \dots, \mathcal{J}(t_{i_s})}{x_{i_1}, \dots, x_{i_s}} \models \exists x \psi \\ &\iff \mathcal{J} \frac{\mathcal{J}(t_0), \dots, \mathcal{J}(t_r)}{x_0, \dots, x_r} \models \exists x \psi \\ &\quad (\text{by } x_i \notin \text{free}(\exists x \psi) \text{ or } x_i = t_i \text{ for } i \neq i_1, \dots, i \neq i_s). \quad \square \end{aligned}$$

2 Exercises

Exercise 2.1. Prove Lemma 1.2.

Exercise 2.2. Let S be finite, i.e., containing finitely many relation symbols, function symbols, and constants. Prove that two **finite** S -structures \mathfrak{A} and \mathfrak{B} are isomorphic if and only if for any S -sentence φ

$$\mathfrak{A} \models \varphi \iff \mathfrak{B} \models \varphi.$$

Exercise 2.3. Let P be a binary relation symbol and f a binary function symbol. Set $x := v_0$, $y := v_1$, $u := v_2$, $v := v_3$, and $w := v_4$. Show that:

(a)

$$\exists x \exists y (P_{xu} \wedge P_{yv}) \frac{u, u, u}{x, y, v} = \exists x \exists y (P_{xu} \wedge P_{yu}).$$

(b)

$$\exists x \exists y (P_{xu} \wedge P_{yv}) \frac{v, f_{uv}}{u, v} = \exists x \exists y (P_{xv} \wedge P_{y f_{uv}}).$$

(c)

$$\exists x \exists y (P_{xu} \wedge P_{yv}) \frac{u, x, f_{uv}}{x, u, v} = \exists w \exists y (P_{wx} \wedge P_{y f_{uv}}).$$

(c)

$$[\forall x \exists y (P_{xy} \wedge P_{xu}) \vee \exists u f_{uu} \equiv x] \frac{x, f_{xy}}{x, u} = \forall v \exists w (P_{vw} \wedge P_{v f_{xy}}) \vee \exists u f_{uu} \equiv x.$$