Mathematical Logic (V)

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1 The Semantics of First-order Logic

1.1 Substitution

Definition 1.1. Let t be an S-term, $x_0, \ldots, x_r$ variables, and $t_0, \ldots, t_r$ S-terms. Then the term

$$t\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r}$$

is defined inductively as follows.

(a) Let $t = x$ be a variable. Then

$$t\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} := \begin{cases} t_i & \text{if } x = x_i \text{ for some } 0 \leqslant i \leqslant r \\ x & \text{otherwise.} \end{cases}$$

(b) For a constant $t = c$

$$c\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} := c.$$

(c) For a function term

$$ft'_1 \ldots t'_n\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} := ft'_1\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} \ldots t'_n\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r}.$$

$\dashv$

Definition 1.2. Let φ be an S-formula, $x_0, \ldots, x_r$ variables, and $t_0, \ldots, t_r$ S-terms. We define

$$\varphi\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r}$$

inductively as follow.

(a) Assume $\varphi = t'_1 \equiv t'_2$. Then

$$\varphi\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} := t'_1\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} \equiv t'_2\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r}.$$

(b) Let $\varphi = Rt'_1 \ldots t'_n$. We set

$$\varphi\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} := Rt'_1\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} \ldots t'_n\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r}.$$

(c) For $\varphi = \neg\psi$

$$\varphi\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r} := \neg\psi\frac{t_0, \ldots, t_r}{x_0, \ldots, x_r}.$$

(d) For $\varphi = (\psi_1 \vee \psi_2)$

$$\varphi\frac{t_0,\ldots,t_r}{x_0,\ldots,x_r} := \left(\psi_1\frac{t_0,\ldots,t_r}{x_0,\ldots,x_r} \vee \psi_2\frac{t_0,\ldots,t_r}{x_0,\ldots,x_r}\right).$$

(e) Assume $\varphi = \exists x\psi$. Let $x_{i_1},\ldots,x_{i_s}$ ($i_1 < \ldots < i_s$) be the variables x_i in $x_0,\ldots,x_r$ with $x_i \in \text{free}(\exists x\varphi)$ and $x_i \neq t_i$. In particular, $x \neq x_{i_1},\ldots,x \neq x_{i_s}$. Then

$$\varphi\frac{t_0,\ldots,t_r}{x_0,\ldots,x_r} := \exists u\left[\psi\frac{t_{i_1},\ldots,t_{i_s},u}{x_{i_1},\ldots,x_{i_s},x}\right],$$

where $u = x$ if x does not occur in $t_{i_1},\ldots,t_{t_s}$; otherwise u is the first variable in $\{v_0,v_1,v_2,\ldots\}$ which does not occur in $\psi, t_{i_1},\ldots,t_{i_s}$. ⊣

Definition 1.3. Let β be an assignment in $\mathfrak{A}$ and $a_0,\ldots,a_r \in A$. Then

$$\beta\frac{a_0,\ldots,a_r}{x_0,\ldots,x_r}$$

is an assignment in $\mathfrak{A}$ defined by

$$\beta\frac{a_0,\ldots,a_r}{x_0,\ldots,x_r}(y) := \begin{cases} a_i & \text{if } y = x_i \text{ for } 0 \leqslant i \leqslant r \\ \beta(y) & \text{otherwise.}\end{cases}$$

For an S-interpretation $\mathfrak{I} = (\mathfrak{A},\beta)$ we let

$$\mathfrak{I}\frac{a_0,\ldots,a_r}{x_0,\ldots,x_r} := \left(\mathfrak{A}, \beta\frac{a_0,\ldots,a_r}{x_0,\ldots,x_r}\right).$$ ⊣

Lemma 1.4 (The Substitution Lemma). *(a) For every S-term* t

$$\mathfrak{I}\left(t\frac{t_0,\ldots,t_r}{x_0,\ldots x_r}\right) = \mathfrak{I}\frac{\mathfrak{I}(t_0),\ldots,\mathfrak{I}(t_r)}{x_0,\ldots x_r}(t).$$

(b) For every S-formula φ

$$\mathfrak{I} \models \varphi\frac{t_0,\ldots,t_r}{x_0,\ldots x_r} \iff \mathfrak{I}\frac{\mathfrak{I}(t_0),\ldots,\mathfrak{I}(t_r)}{x_0,\ldots x_r} \models \varphi.$$ ⊣

Proof: (a) Assume $t = x$. If $x \neq x_i$ for all $0 \leqslant i \leqslant r$, then

$$t\frac{t_0,\ldots,t_r}{x_0,\ldots,x_r} = x.$$

Therefore,

$$\mathfrak{I}\left(t\frac{t_0,\ldots,t_r}{x_0,\ldots,x_r}\right) = \mathfrak{I}(x) = \mathfrak{I}\frac{\mathfrak{I}(t_0),\ldots,\mathfrak{I}(t_r)}{x_0,\ldots,x_r}(x) = \mathfrak{I}\frac{\mathfrak{I}(t_0),\ldots,\mathfrak{I}(t_r)}{x_0,\ldots,x_r}(t).$$

Otherwise, $x = x_i$ for some $0 \leqslant i \leqslant r$. Then $t\frac{t_0,\ldots,t_r}{x_0,\ldots,x_r} = t_i$. It follows that

$$\mathfrak{I}\left(t\frac{t_0,\ldots,t_r}{x_0,\ldots,x_r}\right) = \mathfrak{I}(t_i) = \mathfrak{I}\frac{\mathfrak{I}(t_0),\ldots,\mathfrak{I}(t_r)}{x_0,\ldots,x_r}(x_i) = \mathfrak{I}\frac{\mathfrak{I}(t_0),\ldots,\mathfrak{I}(t_r)}{x_0,\ldots,x_r}(t).$$

The other cases of t can be shown similarly.

(b) Assume that $\varphi = Rt'_1 \dots t'_n$. Then

$$\begin{aligned}
\mathfrak{I} \models \varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r} &\iff \left(\mathfrak{I}\left(t'_1\frac{t_0,\dots,t_r}{x_0,\dots,x_r}\right),\dots,\mathfrak{I}\left(t'_n\frac{t_0,\dots,t_r}{x_0,\dots,x_r}\right)\right) \in R^{\mathfrak{A}} \\
&\iff \left(\mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r}(t'_1),\dots,\mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r}(t'_n)\right) \in R^{\mathfrak{A}} \qquad \text{(by (a))} \\
&\iff \mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r} \models Rt'_1\dots t'_n \\
&\qquad\qquad \text{i.e., } \mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r} \models \varphi.
\end{aligned}$$

For another case, let $\varphi = \exists x\psi$. Again, let $x_{i_1},\dots,x_{i_s}$ be the variables x_i with $x_i \in \text{free}(\exists x\psi)$ and $x_i \neq t_i$. Choose u according to Definition 1.2 (e). In particular, u does not occur in $t_{i_1},\dots,t_{i_s}$. Then

$$\begin{aligned}
\mathfrak{I} \models \varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r} &\iff \mathfrak{I} \models \exists u\left[\psi\frac{t_{i_1},\dots,t_{i_s},u}{x_{i_1},\dots,x_{i_s},x}\right] \\
&\iff \text{there exists an } a \in A \text{ such that } \mathfrak{I}\frac{a}{u} \models \psi\frac{t_{i_1},\dots,t_{i_s},u}{x_{i_1},\dots,x_{i_s},x} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathfrak{I}\frac{a}{u}\right]\frac{\mathfrak{I}\frac{a}{u}(t_{i_1}),\dots,\mathfrak{I}\frac{a}{u}(t_{i_s}),\mathfrak{I}\frac{a}{u}(u)}{x_{i_1},\dots,x_{i_s},x} \models \psi \\
&\qquad\qquad \text{(by induction hypothesis)} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathfrak{I}\frac{a}{u}\right]\frac{\mathfrak{I}(t_{i_1}),\dots,\mathfrak{I}(t_{i_s}),a}{x_{i_1},\dots,x_{i_s},x} \models \psi \\
&\qquad\qquad \text{(by the coincidence lemma and that } u \text{ does not occur in } t_{i_1},\dots t_{i_s}) \\
&\iff \text{there exists an } a \in A \text{ such that } \mathfrak{I}\frac{\mathfrak{I}(t_{i_1}),\dots,\mathfrak{I}(t_{i_s}),a}{x_{i_1},\dots,x_{i_s},x} \models \psi \\
&\qquad\qquad \text{(by (either } u = x \text{ or } u \text{ does not occur in } \psi) \text{ and the coincidence lemma)} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathfrak{I}\frac{\mathfrak{I}(t_{i_1}),\dots,\mathfrak{I}(t_{i_s})}{x_{i_1},\dots,x_{i_s}}\right]\frac{a}{x} \models \psi \\
&\qquad\qquad \text{(since } x \neq x_{i_1},\dots,x \neq x_{i_s}) \\
&\iff \mathfrak{I}\frac{\mathfrak{I}(t_{i_1}),\dots,\mathfrak{I}(t_{i_s})}{x_{i_1},\dots,x_{i_s}} \models \exists x\psi \\
&\iff \mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r} \models \exists x\psi \\
&\qquad\qquad \text{(by } x_i \notin \text{free}(\exists x\psi) \text{ or } x_i = t_i \text{ for } i \neq i_1,\dots,i \neq i_s). \qquad \square
\end{aligned}$$

2 Sequent Calculus

The goal of this section is to provide a formal definition of proofs, i.e., proofs are made into mathematical objects. To that end, we divide any proof into stages. In each stage, we establish a fact that under the **antecedent** $\varphi_1,\dots,\varphi_n$[1] the **succedent** φ holds. In a succinct form, this is written as a sequent

$$\varphi_1 \dots \varphi_n \varphi.$$

So our goal is to design a calculus $\mathfrak{S}$ operating on sequents, i.e., **sequent calculus**. $\mathfrak{S}$ contains a number of rules, which enable us to derive one sequent from another.

[1] In the sequel, we tacitly assume a fixed symbol set S.

Definition 2.1. If in the calculus $\mathfrak{S}$ there is a derivation of the sequent $\Gamma\ \varphi$, then we write

$$\vdash \Gamma\ \varphi$$

and say that $\Gamma\ \varphi$ is **derivable**. $\dashv$

Definition 2.2. A formula φ is **formally provable** or **derivable** from a set Φ of formulas, written $\Phi \vdash \varphi$, if there are finite many formulas $\varphi_1, \ldots, \varphi_n$ in Φ such that

$$\vdash \varphi_1 \ldots \varphi_n\ \varphi.$$ $\dashv$

Definition 2.3. A sequent $\Gamma\ \varphi$ is **correct** if

$$\{\psi \mid \psi \text{ is a member of } \Gamma\} \models \varphi.$$

in short, $\Gamma \models \varphi$. $\dashv$

2.1 Basic Rules

Antecedent

$$\frac{\Gamma \quad \varphi}{\Gamma' \quad \varphi}\ \Gamma \subseteq \Gamma'$$

The correctness is straightforward. Assume that $\Gamma \models \varphi$ and $\mathfrak{I} \models \Gamma'$. Since $\Gamma \subseteq \Gamma'$, we conclude $\mathfrak{I} \models \Gamma$ and thus $\mathfrak{I} \models \varphi$.

Assumption

$$\frac{}{\Gamma \quad \varphi}\ \varphi \in \Gamma$$

Case Analysis

$$\frac{\begin{array}{lll}\Gamma & \psi & \varphi \\ \Gamma & \neg\psi & \varphi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

Contradiction

$$\frac{\begin{array}{lll}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

$\vee$-introduction in antecedent

$$\frac{\begin{array}{rll}\Gamma & \varphi & \chi \\ \Gamma & \psi & \chi\end{array}}{\Gamma \quad (\varphi \vee \psi) \quad \chi}$$

$\vee$-introduction in succedent

$$\text{(a)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad (\varphi \vee \psi)} \qquad\qquad \text{c (b)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad (\psi \vee \varphi)}$$

∃-introduction in succedent

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad \exists x\varphi}$$

∃-introduction in antecedent

$$\frac{\Gamma \quad \varphi\frac{y}{x} \quad \psi}{\Gamma \quad \exists x\varphi \quad \psi} \text{ if } y \notin \text{free}\big(\Gamma \cup \{\exists x\varphi, \psi\}\big)$$

Equality

$$\frac{}{t \equiv t}$$

Substitution

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad t \equiv t' \quad \varphi\frac{t'}{x}}$$

2.2 Some Derived Rules

Example 2.4 (The law of excluded middle).

1.	φ	φ	(assumption)
2.	φ	$(\varphi \vee \neg\varphi)$	($\vee$-introduction in succedent by 1)
3.	$\neg\varphi$	$\neg\varphi$	(assumption)
4.	$\neg\varphi$	$(\varphi \vee \neg\varphi)$	($\vee$-introduction in succedent by 3)
5.		$(\varphi \vee \neg\varphi)$	(case analysis by 2 and 4).

Therefore $\vdash (\varphi \vee \neg\varphi)$. ⊣

Example 2.5 (The modified contradiction).

$$\frac{\begin{array}{ll}\Gamma & \psi \\ \Gamma & \neg\psi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

We argue as follows.

1.	Γ	ψ	(premise)
2.	Γ	$\neg\psi$	(premise)
3.	$\Gamma \quad \neg\varphi$	ψ	(antecedent by 1)
4.	$\Gamma \quad \neg\varphi$	$\neg\psi$	(antecedent by 2)
5.	Γ	φ	(contradiction by 3 and 4).

⊣

Example 2.6 (The chain deduction).

$$\frac{\begin{array}{rl}\Gamma & \varphi \\ \Gamma \quad \varphi & \psi\end{array}}{\begin{array}{rl}\Gamma & \psi\end{array}}$$

We have the following deduction.

1.	Γ		φ	(premise)
2.	Γ	φ	ψ	(premise)
3.	Γ	$\neg\varphi$	φ	(antecedent by 1)
4.	Γ	$\neg\varphi$	$\neg\varphi$	(assumption)
5.	Γ	$\neg\varphi$	ψ	(modified contradiction by 3 and 4)
6.	Γ		ψ	(case analysis by 2 and 5).

$\dashv$

Let Φ be a set of sentences and φ an formula.

Lemma 2.7. $\Phi \vdash \varphi$ *if and only if there exists a* ***finite*** $\Phi_0 \subseteq \Phi$ *such that* $\Phi_0 \vdash \varphi$. $\dashv$

Theorem 2.8 (Soundness)**.** *If* $\Phi \vdash \varphi$*, then* $\Phi \models \varphi$. $\dashv$

3 Exercises

Exercise 3.1. Can you derive the rule of contradiction from the modified contradiction?

Exercise 3.2. Prove:

$$\text{(a)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad \neg\neg\varphi} \qquad\qquad \text{(b)}\ \frac{\Gamma \quad \neg\neg\varphi}{\Gamma \quad \varphi}$$

Exercise 3.3. Is the following derivable?

$$\frac{}{\Gamma \quad \exists x\varphi \qquad \forall x\varphi}$$