# Mathematical Logic (VI)

Yijia Chen

## 1 Sequent Calculus

### 1.1 Basic Rules

**Antecedent**

$$\frac{\Gamma \quad \varphi}{\Gamma' \quad \varphi}\ \Gamma \subseteq \Gamma'$$

The correctness is straightforward. Assume that $\Gamma \models \varphi$ and $\mathfrak{I} \models \Gamma'$. Since $\Gamma \subseteq \Gamma'$, we conclude $\mathfrak{I} \models \Gamma$ and thus $\mathfrak{I} \models \varphi$.

**Assumption**

$$\frac{}{\Gamma \quad \varphi}\ \varphi \in \Gamma$$

**Case Analysis**

$$\frac{\begin{array}{lll}\Gamma & \psi & \varphi \\ \Gamma & \neg\psi & \varphi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

**Contradiction**

$$\frac{\begin{array}{lll}\Gamma & \neg\varphi & \psi \\ \Gamma & \neg\varphi & \neg\psi\end{array}}{\begin{array}{lll}\Gamma & & \varphi\end{array}}$$

**$\vee$-introduction in antecedent**

$$\frac{\begin{array}{lll}\Gamma & \varphi & \chi \\ \Gamma & \psi & \chi\end{array}}{\begin{array}{lll}\Gamma & (\varphi \vee \psi) & \chi\end{array}}$$

**$\vee$-introduction in succedent**

$$\text{(a)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad (\varphi \vee \psi)} \qquad\qquad \text{(b)}\ \frac{\Gamma \quad \varphi}{\Gamma \quad (\psi \vee \varphi)}$$

**$\exists$-introduction in succedent**

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad \exists x \varphi}$$

**∃-introduction in antecedent**

$$\frac{\Gamma \quad \varphi\frac{y}{x} \quad \psi}{\Gamma \quad \exists x\varphi \quad \psi} \text{ if } y \notin \text{free}\big(\Gamma \cup \{\exists x\varphi, \psi\}\big)$$

**Equality**

$$\frac{}{t \equiv t}$$

**Substitution**

$$\frac{\Gamma \quad \varphi\frac{t}{x}}{\Gamma \quad t \equiv t' \quad \varphi\frac{t'}{x}}$$

## 1.2 Some Derived Rules

**Example 1.1** (The law of excluded middle).

| | | | |
|---|---|---|---|
| 1. | $\varphi$ | $\varphi$ | (assumption) |
| 2. | $\varphi$ | $(\varphi \vee \neg\varphi)$ | ($\vee$-introduction in succedent by 1) |
| 3. | $\neg\varphi$ | $\neg\varphi$ | (assumption) |
| 4. | $\neg\varphi$ | $(\varphi \vee \neg\varphi)$ | ($\vee$-introduction in succedent by 3) |
| 5. | | $(\varphi \vee \neg\varphi)$ | (case analysis by 2 and 4). |

Therefore $\vdash (\varphi \vee \neg\varphi)$. ⊣

**Example 1.2** (The modified contradiction).

$$\frac{\begin{array}{ll}\Gamma & \psi \\ \Gamma & \neg\psi\end{array}}{\begin{array}{ll}\Gamma & \varphi\end{array}}$$

We argue as follows.

| | | | |
|---|---|---|---|
| 1. | $\Gamma$ | $\psi$ | (premise) |
| 2. | $\Gamma$ | $\neg\psi$ | (premise) |
| 3. | $\Gamma \quad \neg\varphi$ | $\psi$ | (antecedent by 1) |
| 4. | $\Gamma \quad \neg\varphi$ | $\neg\psi$ | (antecedent by 2) |
| 5. | $\Gamma$ | $\varphi$ | (contradiction by 3 and 4). |

⊣

**Example 1.3** (The chain deduction).

$$\frac{\begin{array}{rl}\Gamma & \varphi \\ \Gamma \quad \varphi & \psi\end{array}}{\begin{array}{rl}\Gamma & \psi\end{array}}$$

We have the following deduction.

| | | | |
|---|---|---|---|
| 1. | $\Gamma$ | $\varphi$ | (premise) |
| 2. | $\Gamma \quad \varphi$ | $\psi$ | (premise) |
| 3. | $\Gamma \quad \neg\varphi$ | $\varphi$ | (antecedent by 1) |
| 4. | $\Gamma \quad \neg\varphi$ | $\neg\varphi$ | (assumption) |
| 5. | $\Gamma \quad \neg\varphi$ | $\psi$ | (modified contradiction by 3 and 4) |
| 6. | $\Gamma$ | $\psi$ | (case analysis by 2 and 5). |

$\dashv$

**Definition 1.4.** Let $\Phi$ be a set of S-formulas and $\varphi$ an S-formula. Then $\varphi$ is **derivable from** $\Phi$, denoted by $\Phi \vdash \varphi$, if there exists an $n \in \mathbb{N}$ and $\varphi_1, \dots, \varphi_n \in \Phi$ such that

$$\vdash \varphi_1 \dots \varphi_n \ \varphi.$$

$\dashv$

Let $\Phi$ be a set of S-sentences and $\varphi$ an S-formula.

**Lemma 1.5.** *$\Phi \vdash \varphi$ if and only if there exists a **finite** $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \vdash \varphi$.* $\dashv$

**Theorem 1.6** (Soundness). *If $\Phi \vdash \varphi$, then $\Phi \models \varphi$.* $\dashv$

## 2 Consistency

**Definition 2.1.** $\Phi$ is **consistent**, written $\text{cons}(\Phi)$, if there is no $\varphi$ such that both $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. Otherwise, $\Phi$ is **inconsistent**.

**Lemma 2.2.** *$\Phi$ is inconsistent if and only if $\Phi \vdash \varphi$ for any formula $\varphi$.*

*Proof:* The direction from right to left is by Definition **??**. For the converse direction, assume that there is a $\psi$ such that $\Phi \vdash \psi$ and $\Phi \vdash \neg\psi$. Then there exist two finite sequences of formulas, $\Gamma_1$ and $\Gamma_2$, such that we have derivation

$$\begin{array}{cc} & \vdots \\ \Gamma_1 & \psi \end{array} \quad \text{and} \quad \begin{array}{cc} & \vdots \\ \Gamma_2 & \neg\psi. \end{array}$$

Then for every $\varphi$ we can obtain the derivation of $\Gamma_1\ \Gamma_2\ \varphi$ as below.

$$\begin{array}{rcccr}
 & & \vdots & & \\
m. & & \Gamma_1 & \psi & \\
 & & \vdots & & \\
n. & & \Gamma_2 & \neg\psi & \\
(n+1). & \Gamma_1 & \Gamma_2 & \psi & \text{(antecedent by } m) \\
(n+2). & \Gamma_1 & \Gamma_2 & \neg\psi & \text{(antecedent by } n) \\
(n+3). & \Gamma_1 & \Gamma_2 & \varphi & \text{(modified contradiction by } n+1 \text{ and } n+2).
\end{array}$$

$\Box$

**Corollary 2.3.** *$\Phi$ is consistent if and only if there is a $\varphi$ such that $\Phi \nvdash \varphi$.*

**Lemma 2.4.** *$\Phi$ is consistent if and only if every finite $\Phi_0 \subseteq \Phi$ is consistent.*

**Lemma 2.5.** *Every satisfiable $\Phi$ is consistent.*

*Proof:* Assume that $\Phi$ is inconsistent. Then there is a $\varphi$ such that $\Phi \vdash \varphi$ and $\Phi \vdash \neg\varphi$. By the Soundness Theorem, i.e., Theorem **??**, we conclude $\Phi \models \varphi$ and $\Phi \models \neg\varphi$. Thus, $\Phi$ cannot be satisfiable. $\Box$

**Lemma 2.6.** *(a) $\Phi \vdash \varphi$ if and only if $\Phi \cup \{\neg\varphi\}$ is inconsistent.*

*(b) $\Phi \vdash \neg\varphi$ if and only if $\Phi \cup \{\varphi\}$ is inconsistent.*

*(c) If $\text{cons}(\Phi)$, then either $\text{cons}\big(\Phi \cup \{\varphi\}\big)$ or $\text{cons}(\Phi \cup \{\neg\varphi\})$.*

# 3 Completeness

The goal of this section is to show:

**Theorem 3.1** (Completeness)**.** *If* $\Phi \models \varphi$*, then* $\Phi \vdash \varphi$*.* $\dashv$

We observe that the contrapositive of Theorem **??** is:

$$\Phi \nvdash \varphi \text{ implies } \Phi \not\models \varphi$$
$$\iff \text{if } \Phi \cup \{\neg\varphi\} \text{ is consistent, then } \Phi \cup \{\neg\varphi\} \text{ is satisfiable.}$$

As a matter of fact, we actually will prove the following general statement.

**Theorem 3.2.** $\text{cons}(\Phi)$ *implies that* $\Phi$ *is satisfiable.* $\dashv$

## 3.1 Henkin's Theorem

We fix a set $\Phi$ of S-formulas and will construct an S-interpretation out of $\Phi$. To that end, we first define a binary relation over the set $T^S$ of S-terms.

**Definition 3.3.** Let $t_1, t_2 \in T^S$. Then $t_1 \sim t_2$ if $\Phi \vdash t_1 \equiv t_2$. $\dashv$

**Lemma 3.4.** *(i)* $\sim$ *is an* ***equivalence*** *relation.*

*(ii)* $\sim$ *is a* ***congruence*** *relation. That is:*

- *For every* $n$*-ary function symbol* $f \in S$ *and* $2 \cdot n$ *S-terms* $t_1 \sim t'_1, \ldots, t_n \sim t'_n$*, we have*

$$ft_1 \cdots t_n \sim ft'_1 \cdots t'_n.$$

- *For every* $n$*-ary relation symbol* $R \in S$ *and* $2 \cdot n$ *S-terms* $t_1 \sim t'_1, \ldots, t_n \sim t'_n$*, we have*

$$\Phi \vdash Rt_1 \cdots t_n \quad \iff \quad \Phi \vdash Rt'_1 \cdots t'_n.$$

$\dashv$

*Proof:* By the equality rule and the substitution rule. □

Now for every $t \in T^S$ we define

$$\bar{t} := \{t' \in T^S \mid t' \sim t\},$$

i.e., the equivalence class of $t$.

**Definition 3.5.** The **term structure for** $\Phi$, denoted by $\mathfrak{T}^\Phi$, is defined as follows.

(i) The universe is $T^\Phi := \{\bar{t} \mid t \in T^S\}$.

(ii) For every $n$-ary relation symbol $R \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$(\bar{t}_1, \ldots, \bar{t}_n) \in R^{\mathfrak{T}^\Phi} \quad \text{if} \quad \Phi \vdash Rt_1 \ldots t_n.$$

(iii) For every $n$-ary function symbol $f \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$f^{\mathfrak{T}^\Phi}(\bar{t}_1, \ldots, \bar{t}_n) := \overline{ft_1 \cdots t_n}.$$

(iv) For every constant $c \in S$

$$c^{\mathfrak{T}^\Phi} := \bar{c}.$$

This finishes the construction of $\mathfrak{T}^\Phi$. ⊣

Using Lemma **??**, in particular (ii), it is easy to verify that:

**Lemma 3.6.** *$\mathfrak{T}^\Phi$ is well-defined.* ⊣

To complete the definition of an S-interpretation, we still need to provide an assignment of the variables $v_0, v_1, \dots$ in $\mathfrak{T}^\Phi$.

**Definition 3.7.** For every variable $v_i$ we let

$$\beta^\Phi(v_i) := \bar{v}_i.$$ ⊣

Finally we let

$$\mathfrak{I}^\Phi := \left(\mathfrak{T}^\Phi, \beta^\Phi\right).$$

**Lemma 3.8.** *(i) For any* $t \in T^S$

$$\mathfrak{I}^\Phi(t) = \bar{t}.$$

*(ii) For every **atomic** $\varphi$*

$$\mathfrak{I}^\Phi \models \varphi \quad \Longleftrightarrow \quad \Phi \vdash \varphi.$$

*Proof:* (i) We proceed by induction on t.

- $t = v_i$ is a variable. Then
$$\mathfrak{I}^\Phi(v_i) = \beta^\Phi(v_i) = \bar{v}_i.$$
- $t = c$ is a constant. Then
$$\mathfrak{I}^\Phi(c) = c^{\mathfrak{T}^\Phi} = \bar{c}$$
- $t = ft_1 \cdots t_n$. Then
$$\begin{aligned} \mathfrak{I}^\Phi(ft_1\cdots t_n) &= f^{\mathfrak{T}^\Phi}(\mathfrak{I}^\Phi(t_1), \dots, \mathfrak{I}^\Phi(t_n)) \\ &= f^{\mathfrak{T}^\Phi}(\bar{t}_1, \dots, \bar{t}_n) && \text{(by induction hypothesis)} \\ &= \overline{ft_1\cdots t_n}. \end{aligned}$$

(ii) Recall that there are two types of atomic formulas. For the first type, let $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned} \mathfrak{I}^\Phi \models t_1 \equiv t_2 &\iff \mathfrak{I}^\Phi(t_1) = \mathfrak{I}^\Phi(t_2) \\ &\iff \bar{t}_1 = \bar{t}_2 && \text{(by (i))} \\ &\iff t_1 \sim t_2 \\ &\iff \Phi \vdash t_1 \equiv t_2. \end{aligned}$$

Second, let $\varphi = Rt_1 \cdots t_n$. We deduce

$$\begin{aligned} \mathfrak{I}^\Phi \models Rt_1\cdots t_n &\iff \left(\mathfrak{I}^\Phi(t_1), \dots, \mathfrak{I}^\Phi(t_n)\right) \in R^{\mathfrak{T}^\Phi} \\ &\iff \left(\bar{t}_1, \dots, \bar{t}_n\right) \in R^{\mathfrak{T}^\Phi} && \text{(by (i))} \\ &\iff \Phi \vdash Rt_1\cdots t_n. \end{aligned}$$

□

# 4 Exercises

**Exercise 4.1.** Prove Lemma **??**

**Exercise 4.2.** Let

$$\Phi := \left\{\forall x \neg Rxx, \forall x \forall y \forall z\big(Rxy \land Ryz) \to Rxz\big), \forall x \forall y\big(x \equiv y \lor Rxy \lor Ryx\big), \forall x \exists y Rxy\right\}.$$

Prove that $\Phi$ is consistent.

**Exercise 4.3.** Let $S := \{R\}$ with unary relation symbol $R$. Moreover we define

$$\Phi := \{\exists x Rx\} \cup \{\neg Ry \mid \text{for every variable } y\}.$$

Prove that:

- $\Phi$ is consistent.
- There is no term $t \in T^S$ with $\Phi \vdash Rt$.