# Mathematical Logic (XIII)

Yijia Chen

## 1. Gödel's Incompleteness Theorems

Let $\mathbb{P}$ be a program over $\mathcal{A}$. Assume that $\mathbb{P}$ consists of instructions $\alpha_0, \ldots, \alpha_k$. Let $n$ be the maximum index $i$ such that $R_i$ is used by $\mathbb{P}$. Then a configuration of $\mathbb{P}$ is an $(n+2)$-tuple

$$(L, m_0, \ldots, m_n),$$

where $L \leqslant k$ and $m_0, \ldots, m_n \in \mathbb{N}$, meaning that $\alpha_L$ is the instruction to be executed next and every register $R_i$ contains $m_i$, i.e., the word $\underbrace{|\,|\cdots|}_{m_i \text{ times}}$.

We have shown:

**Lemma 1.1.** *From the above program $\mathbb{P}$ we can compute an $S_{ar}$-formula*

$$\chi_{\mathbb{P}}(x_0, \ldots, x_n, z, y_0, \ldots, y_n)$$

*such that for all $\ell_0, \ldots, \ell_n, L, m_0, \ldots, m_n \in \mathbb{N}$*

$$\mathfrak{N} \models \chi_{\mathbb{P}}[\ell_0, \ldots, \ell_n, L, m_0, \ldots, m_n]$$

*if and only if $\mathbb{P}$, beginning with the configuration $(0, \ell_0, \ldots, \ell_n)$, after finitely many steps, reaches the configuration $(L, m_0, \ldots, m_n)$.* ⊣

Using Lemma 1.1 it is now routine to prove:

**Theorem 1.2.** *Let $r \geqslant 1$.*

*(i) Let $\mathscr{R} \subseteq \mathbb{N}^r$ be an R-decidable relation. Then there is an $L^{S_{ar}}$-formula $\varphi(v_0, \ldots, v_{r-1}) \in \mathbb{N}$ such that for all $\ell_0, \ldots, \ell_{r-1} \in \mathbb{N}$*

$$\big(\ell_0, \ldots, \ell_{r-1}\big) \in \mathscr{R} \quad \iff \quad \mathfrak{N} \models \varphi(\bar{\ell}_0, \ldots, \bar{\ell}_{r-1}).$$

*(ii) Let $f : \mathbb{N}^r \to \mathbb{N}$ be an R-computable function. Then there is an $L^{S_{ar}}$-formula $\varphi(v_0, \ldots, v_{r-1}, v_r)$ such that for all $\ell_0, \ldots, \ell_{r-1}, \ell_r \in \mathbb{N}$*

$$f\big(\ell_0, \ldots, \ell_{r-1}\big) = \ell_r \quad \iff \quad \mathfrak{N} \models \varphi(\bar{\ell}_0, \ldots, \bar{\ell}_{r-1}, \bar{\ell}_r).$$

*Therefore,*

$$\mathfrak{N} \models \exists^{=1} v_r\ \varphi(\bar{\ell}_0, \ldots, \bar{\ell}_{r-1}, v_r),$$

*where $\exists^{=1} x\ \theta(x)$ denotes the formula*

$$\exists x \Big(\theta(x) \land \forall y \big(\theta(y) \to y \equiv x\big)\Big).$$ ⊣

Let $\Phi \subseteq L_0^{S_{ar}}$.

**Definition 1.3.** Let $r \geqslant 1$.

(i) A relation $\mathscr{R} \subseteq \mathbb{N}^r$ is *representable in* $\Phi$ if there is an $L^{S_{ar}}$-formula $\varphi(v_0, \ldots, v_{r-1})$ such that for all $n_0, \ldots, n_{r-1} \in \mathbb{N}$

$$\begin{aligned}(n_0, \ldots, n_{r-1}) \in \mathscr{R} &\implies \Phi \vdash \varphi(\bar{n}_0, \ldots, \bar{n}_{r-1}),\\ (n_0, \ldots, n_{r-1}) \notin \mathscr{R} &\implies \Phi \vdash \neg\varphi(\bar{n}_0, \ldots, \bar{n}_{r-1}).\end{aligned}$$

(ii) A function $F : \mathbb{N}^r \to \mathbb{N}$ is *representable in* $\Phi$ if there is an $L^{S_{ar}}$-formula $\varphi(v_0, \ldots, v_{r-1}, v_r)$ such that for all $n_0, \ldots, n_{r-1}, n_r \in \mathbb{N}$

$$\begin{aligned}f(n_0, \ldots, n_{r-1}) = n_r &\implies \Phi \vdash \varphi(\bar{n}_0, \ldots, \bar{n}_{r-1}, \bar{n}_r),\\ f(n_0, \ldots, n_{r-1}) \neq n_r &\implies \Phi \vdash \neg\varphi(\bar{n}_0, \ldots, \bar{n}_{r-1}, \bar{n}_r).\end{aligned}$$

Moreover,

$$\Phi \vdash \exists^{=1} v_r \; \varphi(\bar{n}_0, \ldots, \bar{n}_{r-1}, v_r). \qquad \dashv$$

**Lemma 1.4.** *(i) If $\Phi$ is inconsistent, then every relation over $\mathbb{N}$ and every function over $\mathbb{N}$ is representable in $\Phi$.*

*(ii) Let $\Phi \subseteq \Phi' \subseteq L_0^{S_{ar}}$. Then every relation representable in $\Phi$ is also representable in $\Phi'$. Similarly, every function representable in $\Phi$ is representable in $\Phi'$ as well.*

*(iii) Let $\Phi$ be consistent. If $\Phi$ is R-decidable, then every relation representable in $\Phi$ is R-decidable, and every function representable in $\Phi$ is R-computable.* $\dashv$

**Definition 1.5.** $\Phi$ *allows representations* if all R-decidable relations and all R-computable functions over $\mathbb{N}$ are representable in $\Phi$. $\dashv$

By Theorem 1.2:

**Theorem 1.6.** $\mathrm{Th}(\mathfrak{N})$ *allows representations.* $\dashv$

With some extra efforts we can prove:

**Theorem 1.7.** $\Phi_{\mathrm{PA}}$ *allows representations.* $\dashv$

Recall that we have exhibited the so-called Gödel numbering of register programs. For later purposes, we do the same for $L^{S_{ar}}$-formulas. Let

$$\varphi_0, \varphi_1, \ldots, \tag{1}$$

be an *effective* enumeration of all $L^{S_{ar}}$-formulas without repetition. That is, there is a program that prints out the sequence (1). Then for every $\varphi \in L^{S_{ar}}$ we let

$$[\varphi] := n \quad \text{where } \varphi = \varphi_n.$$

Observe that both

$$n \mapsto \varphi_n \text{ and } \varphi \mapsto [\varphi]$$

are R-computable.

**Theorem 1.8** (Fixed Point Theorem)**.** *Assume that $\Phi$ allows representations. Then for every $\psi \in L_1^{S_{ar}}$, there is an $S_{ar}$-sentence $\varphi$ such that*

$$\Phi \vdash \varphi \leftrightarrow \psi(\overline{[\varphi]}). \tag{2}$$

*Proof:* We define a function $F : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ as follows. For every $n, m \in \mathbb{N}$

$$F(n,m) := \begin{cases} \left[\varphi_n(\bar{m})\right] & \text{if } \mathrm{free}(\varphi_n) = \{v_0\}, \\ & \quad \text{i.e., } \varphi_n \in L_1^{S_{ar}} \setminus L_0^{S_{ar}}, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that F is R-computable, and for every $\varphi \in L_1^{S_{ar}} \setminus L_0^{S_{ar}}$ we have

$$F([\varphi], m) = \left[\varphi(\bar{m})\right]. \tag{3}$$

Since $\Phi$ allows representations, there is an $S_{ar}$-formula $\varphi_F(x,y,z)$ such that for all $n, m, \ell \in \mathbb{N}$

$$F(n,m) = \ell \implies \Phi \vdash \varphi_F(\bar{n}, \bar{m}, \bar{\ell}), \tag{4}$$

$$F(n,m) \neq \ell \implies \Phi \vdash \neg\varphi_F(\bar{n}, \bar{m}, \bar{\ell}). \tag{5}$$

Moreover,

$$\Phi \vdash \exists^{=1} z\ \varphi_F(\bar{n}, \bar{m}, z). \tag{6}$$

Let

$$\chi(v_0) := \forall x\big(\varphi_F(v_0, v_0, x) \to \psi(x)\big).$$

In particular, $\mathrm{free}(\chi) = \{v_0\}$. Finally we define the desired

$$\varphi := \chi(\bar{n}) \quad \text{with } n = [\chi].$$

We show that (2) holds. First, by (3)

$$F(n,n) = F([\chi], n) = \left[\chi(\bar{n})\right] = [\varphi].$$

Then (4) implies

$$\Phi \vdash \varphi_F\big(\bar{n}, \bar{n}, \overline{[\varphi]}\big) \tag{7}$$

Recall

$$\varphi = \chi(\bar{n}) = \forall x\big(\varphi_F(\bar{n}, \bar{n}, x) \to \psi(x)\big).$$

Combined with (7) we obtain

$$\Phi \cup \{\varphi\} \vdash \psi\big(\overline{[\varphi]}\big).$$

Equivalently

$$\Phi \vdash \varphi \to \psi\big(\overline{[\varphi]}\big).$$

For the other direction in (2), observe that (6) and (7) guarantee that

$$\Phi \vdash \forall z\big(\varphi_F(\bar{n}, \bar{n}, z) \to z \equiv \overline{[\varphi]}\big).$$

Thus

$$\Phi \cup \left\{\psi\big(\overline{[\varphi]}\big)\right\} \vdash \forall x\big(\varphi_F(\bar{n}, \bar{n}, x) \to \psi(x)\big),$$

i.e., $\Phi \cup \left\{\psi\big(\overline{[\varphi]}\big)\right\} \vdash \varphi$. It follows that

$$\Phi \vdash \psi\big(\overline{[\varphi]}\big) \to \varphi. \qquad \square$$

**Definition 1.9.** Let $\Phi \subseteq L^{S_{ar}}$. Then

$$\Phi^{\vdash} := \left\{\varphi \in L^{S_{ar}} \;\middle|\; \Phi \vdash \varphi\right\}.$$

We say that $\Phi^{\vdash}$ is *representable in* $\Phi$ if

$$\left\{[\varphi] \in \mathbb{N} \;\middle|\; \varphi \in \Phi^{\vdash}\right\} = \left\{[\varphi] \;\middle|\; \varphi \in L^{S_{ar}} \text{ and } \Phi \vdash \varphi\right\}.$$

is representable in $\Phi$. $\dashv$

**Lemma 1.10.** *Let $\Phi \subseteq L^{S_{ar}}$ be consistent and allow representations. Then $\Phi^{\vdash}$ is* not *representable in $\Phi$.*

*Proof:* Assume that $\Phi^{\vdash}$ is representable in $\Phi$. In particular, there is a $\chi(v_0) \in L_1^{S_{ar}}$ such that for all $\varphi \in L_0^{S_{ar}}$

$$\begin{aligned} \Phi \vdash \varphi &\implies \Phi \vdash \chi(\overline{[\varphi]}), \\ \Phi \nvdash \varphi &\implies \Phi \vdash \neg\chi(\overline{[\varphi]}). \end{aligned}$$

Since $\Phi$ is consistent, we conclude

$$\Phi \nvdash \varphi \iff \Phi \vdash \neg\chi(\overline{[\varphi]}). \tag{8}$$

We apply the Fixed Point Theorem 1.8 to $\neg\chi$ to obtain a sentence $\varphi$ such that

$$\Phi \vdash \varphi \leftrightarrow \neg\chi(\overline{[\varphi]}). \tag{9}$$

Then

$$\begin{aligned} \Phi \vdash \varphi &\iff \Phi \vdash \neg\chi(\overline{[\varphi]}) && \text{(by (9))} \\ &\iff \Phi \nvdash \varphi, && \text{(by (8))} \end{aligned}$$

which is a contradiction. □

**Theorem 1.11** (Tarski's Undefinability of the Arithmetic Truth)**.**

*(i) Let $\Phi \subseteq L^{S_{ar}}$ be consistent and allow representations. Then $\Phi^{\models}$ is not representable in $\Phi$.*

*(ii)* $\mathrm{Th}(\mathfrak{N})$ *is not representable in* $\mathrm{Th}(\mathfrak{N})$.

*Proof:* By the Completeness Theorem

$$\Phi^{\models} = \Phi^{\vdash}.$$

So (i) is a direct consequence of Lemma 1.10.

(ii) is a special case of (i). □

**Theorem 1.12** (Gödel's First Incompleteness Theorem)**.** *Let $\Phi \subseteq L^{S_{ar}}$ be consistent and allow representations. Moreover, $\Phi$ is R-decidable. Then there is an $L^{S_{ar}}$-sentence $\varphi$ such that neither $\Phi \vdash \varphi$ nor $\Phi \vdash \neg\varphi$.*

*Proof:* Assume for every $L^{S_{ar}}$-sentence $\varphi$ either $\Phi \vdash \varphi$ or $\Phi \vdash \neg\varphi$. Thus $\Phi$ is complete. By the R-decidability of $\Phi$, we can then conclude that $\Phi^{\vdash}$ is R-decidable too.

Since $\Phi$ allows representations, $\Phi^{\vdash}$ is representable in $\Phi$. Together with the consistency of $\Phi$, we obtain a contradiction to Lemma 1.10. □

In the following we fix an R-decidable $\Phi \subseteq L_0^{S_{ar}}$ which allows representations.

We choose an effective enumeration of all derivations in the sequent calculus associated with $S_{ar}$ and define a relation $\mathscr{H} \subseteq \mathbb{N}^2$ by

$$\begin{aligned} (n, m) \in \mathscr{H} \iff &\text{the } m\text{-th derivation in the above enumeration ends with a sequent} \\ &\psi_0, \ldots, \psi_{k-1}, \varphi \text{ with } \psi_0, \ldots, \psi_{k-1} \in \Phi \text{ and } n = [\varphi], \end{aligned}$$

Clearly, $\mathscr{H}$ is R-decidable by the R-decidability of $\Phi$. Moreover, for every $\varphi \in L^{S_{ar}}$

$$\Phi \vdash \varphi \iff \text{there is an } m \in \mathbb{N} \text{ with } ([\varphi], m) \in \mathscr{H}.$$

Since $\Phi$ allows representation, there is a $\varphi_{\mathscr{H}}(v_0, v_1) \in L_2^{S_{ar}}$ such that for every $n, m \in \mathbb{N}$

$$(n, m) \in \mathscr{H} \implies \Phi \vdash \varphi_{\mathscr{H}}(\bar{n}, \bar{m}), \tag{10}$$
$$(n, m) \notin \mathscr{H} \implies \Phi \vdash \neg\varphi_{\mathscr{H}}(\bar{n}, \bar{m}). \tag{11}$$

We set

$$\mathrm{DER}_{\Phi}(x) := \exists y \varphi_{\mathscr{H}}(x, y),$$

which intuitively says that $x$ is provable in $\Phi$.

Applying Lemma 1.8 to $\psi(x) := \neg\mathrm{DER}_{\varphi}(x)$, we obtain an $L_0^{S_{ar}}$-sentence $\varphi$ such that

$$\Phi \vdash \varphi \leftrightarrow \neg\mathrm{DER}_{\varphi}\big(\overline{[\varphi]}\big). \tag{12}$$

**Lemma 1.13.** *If $\Phi$ is consistent, then $\Phi \nvdash \varphi$.*

*Proof:* Assume that $\Phi \vdash \varphi$, which is given by the $m$-th derivation for some $m \in \mathbb{N}$. In other words,

$$\big([\varphi], m\big) \in \mathscr{H}.$$

Then, (10) implies

$$\Phi \vdash \varphi_{\mathscr{H}}\big(\overline{[\varphi]}, \bar{m}\big).$$

It follows that

$$\Phi \vdash \mathrm{DER}_{\Phi}\big(\overline{[\varphi]}\big).$$

By (12)

$$\Phi \vdash \neg\varphi.$$

Thus $\Phi$ is inconsistent. □

Observe that $\Phi \vdash 0 \equiv 0$, therefore

$$\Phi \text{ is consistent} \iff \Phi \nvdash \neg 0 \equiv 0.$$

Hence,

$$\mathrm{CONS}_{\Phi} := \neg\mathrm{DER}\big([\neg 0 \equiv 0]\big)$$

expresses that $\Phi$ is consistent.

**Lemma 1.14.** *Assume $\Phi_{PA} \subseteq \Phi$. Then*

$$\Phi \vdash \mathrm{CONS}_{\Phi} \to \neg\mathrm{DER}_{\Phi}\big([\varphi]\big),$$

*where $\varphi$ is the sentence in* (12).

*Proof:* A tedious analysis shows that the proof of Lemma 1.13 can be carried out on the basis of $\Phi_{PA}$. □

**Theorem 1.15** (Gödel's Second Incompleteness Theorem). *Assume $\Phi$ is consistent and* R*-decidable with $\Phi_{PA} \subseteq \Phi$. Then*

$$\Phi \nvdash \mathrm{CONS}_{\Phi}.$$

*Proof:* Assume $\Phi \vdash \mathrm{CONS}_{\Phi}$. Then Lemma 1.14 implies

$$\Phi \vdash \neg\mathrm{DER}_{\Phi}\big([\varphi]\big).$$

By (12) we have

$$\Phi \vdash \varphi,$$

which contradicts Lemma 1.13. □