Mathematical Logic (II)

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1 The Syntax of First-order Logic

Example 1.1 (Group Theory).

(G1) For all x, y, z we have $(x \circ y) \circ z = x \circ (y \circ z)$.

(G2) For all x we have $x \circ e = e$.

(G3) For every x there is a y such that $x \circ y = e$.

A group is a triple $\mathfrak{G} = (G, \circ^{\mathfrak{G}}, e^{\mathfrak{G}})$, i.e., a structure $\mathfrak{G}$, which satisfies (G1)–(G3). ⊣

Example 1.2 (Equivalence Relations).

(E1) For all x we have $(x, x) \in R$.

(E2) For all x and y if $(x, y) \in R$ then $(y, x) \in R$.

(E3) For all x, y, z if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$.

An equivalence relation is specified by a structure $\mathfrak{A} = (A, R^{\mathfrak{A}})$ in which R^{A} satisfies (E1)–(E3). ⊣

1.1 Alphabets

Definition 1.3. An **alphabet** is a nonempty set of **symbols**. ⊣

Definition 1.4. Let $\mathbb{A}$ be an alphabet. Then a **word** w over $\mathbb{A}$ is a finite sequence of symbols in $\mathbb{A}$, i.e.,

$$w = w_1 w_2 \cdots w_n$$

where $n \in \mathbb{N}$ and $w_i \in \mathbb{A}$ for every $i \in [n] = \{1, \ldots, n\}$. In case $n = 0$, then w is the **empty word**, denoted by ε. The **length** $|w|$ of w is n. In particular, $|\varepsilon| = 0$.
$\mathbb{A}^*$ denotes the set of all words over $\mathbb{A}$, or equivalently

$$\mathbb{A}^* = \bigcup_{n \in \mathbb{N}} A^n = \bigcup_{n \in \mathbb{N}} \left\{ w_1 \ldots w_n \;\middle|\; w_1, \ldots, w_n \in \mathbb{A} \right\}.$$

⊣

Countable sets

Later on, we will need to count the number of words over a given alphabet.

Definition 1.5. A set M is **countable** if there exists an **injective** function α from $\mathbb{N}$ **onto** M, i.e., $\alpha : \mathbb{N} \to M$ is a bijection. Thereby, we can write

$$M = \left\{ \alpha(n) \;\middle|\; n \in \mathbb{N} \right\} = \left\{ \alpha(0), \alpha(1), \ldots, \alpha(n), \ldots \right\}.$$

A set M is **at most countable** if M is either finite or countable. ⊣

Lemma 1.6. *Let M be a non-empty set. Then the following are equivalent.*

(a) M *is at most countable.*

(b) *There is a surjective function* $f : \mathbb{N} \to M$.

(c) *There is an injective function* $f : M \to \mathbb{N}$. $\dashv$

Lemma 1.7. *Let* $\mathbb{A}$ *be an alphabet which is at most countable. Then* $\mathbb{A}^*$ *is countable.* $\dashv$

1.2 The alphabet of a first-order language

Definition 1.8. The **alphabet of a first-order language** consists of the following symbols.

(a) $v_0, v_1, \ldots$ (variables).

(b) $\neg, \wedge, \vee, \to, \leftrightarrow$, (negation, conjunction, disjunction, implication, if and only if).

(c) $\forall, \exists$, (for all, exists).

(d) $\equiv$, (equality).

(e) $(,)$, (parentheses).

(f) (1) For every $n \geqslant 1$ a set of n**-ary relation symbols**.
  (2) For every $n \geqslant 1$ a set of n**-ary function symbols**.
  (3) A set of **constants**.

Note any set in (f) can be empty. $\dashv$

We use $\mathbb{A}$ to denote the set of symbols in (a)–(e), i.e., the set of **logic symbols**, while S is the set of remaining symbols in (f). Then a first-order language has

$$\mathbb{A}_S := \mathbb{A} \cup S$$

as its alphabet and S as its **symbol set**.

Thus every first-order language has the same set $\mathbb{A}$ of logic symbols but might have different symbol set S.

1.3 Terms and formulas

Throughout this section, we fix a symbol set S.

Definition 1.9. The set T^S of S**-terms** contains precisely those words in $\mathbb{A}_S^*$ which can be obtained by applying the following rules finitely many times.

(T1) Every variable is an S-term.

(T2) Every constant in S is an S-term.

(T3) If $t_1, \ldots, t_n$ are S-terms and f is a n-ary function symbol in S, then $ft_1 \ldots t_n$ is an S-term. $\dashv$

Definition 1.10. The set L^S of S**-formulas** contains precisely those words in $\mathbb{A}_S^*$ which can be obtained by applying the following rules finitely many times.

(A1) Let t_1 and t_2 be two S-terms. Then $t_1 \equiv t_2$ is an S-formula.

(A2) Let $t_1, \ldots, t_n$ be S-terms and R an n-ary relation symbol in S. Then $Rt_1 \cdots t_n$ is also an S-formula.

(A3) If φ is an S-formula, then so is $\neg\varphi$.

(A4) If φ and ψ are S-formulas, then so is $(\varphi * \psi)$ where $* \in \{\wedge, \vee, \to, \leftrightarrow\}$.

(A5) Let φ be an S-formula and x a variable. Then $\forall x\varphi$ and $\exists x\varphi$ are S-formulas, too.

The formulas in (A1) and (A2) are **atomic**, as they don't contain any other S-formulas as subformulas.

- $\neg\varphi$ is the **negation** of φ.
- $(\varphi \wedge \psi)$ is the **conjunction** of φ and ψ.
- $(\varphi \vee \psi)$ is the **disjunction** of φ and ψ.
- $(\varphi \to \psi)$ is the **implication** from φ to ψ.
- $(\varphi \leftrightarrow \psi)$ is the **equivalence** between φ and ψ. $\dashv$

Lemma 1.11. *Let* S *be at most countable. Then both* T^S *and* L^S *are countable.*

Definition 1.12. Let t be an S-term. Then $\mathrm{var}(t)$ is the set of variables in t. Or inductively,

$$\begin{aligned} \mathrm{var}(x) &:= \{x\}, \\ \mathrm{var}(c) &:= \emptyset, \\ \mathrm{var}(ft_1 \dots t_n) &:= \bigcup_{i \in [n]} \mathrm{var}(t_i). \end{aligned}$$ $\dashv$

Definition 1.13. Let φ be an S-formula and x a variable. We say that **an occurrence of** x **in** φ **is free** if it is not in the scope of any $\forall x$ or $\exists x$. Otherwise, the occurrence is **bound**.

Definition 1.14. Let φ be an S-formula. Then $\mathrm{free}(\varphi)$ is the set of variables which have free occurrences in φ. Or inductively,

$$\begin{aligned} \mathrm{free}(t_1 \equiv t_2) &:= \mathrm{var}(t_1) \cup \mathrm{var}(t_2), \\ \mathrm{free}(Rt_1 \cdots t_n) &:= \bigcup_{i \in [n]} \mathrm{var}(t_i), \\ \mathrm{free}(\neg\varphi) &:= \mathrm{free}(\varphi), \\ \mathrm{free}(\varphi * \psi) &:= \mathrm{free}(\varphi) \cup \mathrm{free}(\psi) \quad \text{with } * \in \{\wedge, \vee, \to, \leftrightarrow\}, \\ \mathrm{free}(\forall x\varphi) &:= \mathrm{free}(\varphi) \setminus \{x\}, \\ \mathrm{free}(\exists x\varphi) &:= \mathrm{free}(\varphi) \setminus \{x\}. \end{aligned}$$ $\dashv$

Example 1.15. The formula below shows that a variable might have both free and bound occurrences in the same formula.

$$\begin{aligned} \mathrm{free}((Rxy \to \forall y \neg y \equiv z)) &= \mathrm{free}(Rxy) \cup \mathrm{free}(\forall y \neg y \equiv z) \\ &= \{x, y\} \cup \big(\mathrm{free}(y \equiv z) \setminus \{y\}\big) = \{x, y, z\}. \end{aligned}$$ $\dashv$

Definition 1.16. An S-formula is an S**-sentence** if $\mathrm{free}(\varphi) = \emptyset$. $\dashv$

Recall that **actual** variables we can use are $v_0, v_1, \dots$.

Definition 1.17. Let $n \in \mathbb{N}$. Then

$$L_n^S := \big\{\varphi \bigm| \varphi \text{ an S-formula with } \mathrm{free}(\varphi) \subseteq \{v_0, \dots, v_{n-1}\}\big\}.$$

In particular, L_0^S is the set of S-sentences. $\dashv$

2 The Semantics of First-order Logic

2.1 Structures and interpretations

We fix a symbol set S.

Definition 2.1. An S-**structure** is a pair $\mathfrak{A} = (A, \mathfrak{a})$ which satisfies the following conditions.

1. $A \neq \emptyset$ is the **universe** of $\mathfrak{A}$.
2. $\mathfrak{a}$ is a function defined on S such that:
   (a) Let $R \in S$ be an n-ary relation symbol. Then $\mathfrak{a}(R) \subseteq A^n$.
   (b) Let $f \in S$ be an n-ary function symbol. Then $\mathfrak{a}(f) : A^n \to A$.
   (c) $\mathfrak{a}(c) \in A$ for every constant $c \in S$.

For better readability, we write $R^{\mathfrak{A}}$, $f^{\mathfrak{A}}$, and $c^{\mathfrak{A}}$, or even R^A, f^A, and c^A, instead of $\mathfrak{a}(R)$, $\mathfrak{a}(f)$, and $\mathfrak{a}(c)$. Thus for $S = \{R, f, c\}$ we might write an S-structure as

$$\mathfrak{A} = \left(A, R^{\mathfrak{A}}, f^{\mathfrak{A}}, c^{\mathfrak{A}}\right) = \left(A, R^A, f^A, c^A\right). \qquad \dashv$$

Examples 2.2. 1. For $S_{\mathrm{Ar}} := \{+, \cdot, 0, 1\}$ the S_{Ar}-structure

$$\mathfrak{N} = \left(\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}\right)$$

is the standard model of natural numbers with addition, multiplication, and constants 0 and 1.

2. For $S_{\mathrm{Ar}}^< := \{+, \cdot, 0, 1, <\}$ we have an $S_{\mathrm{Ar}}^<$-structure

$$\mathfrak{N}^< = \left(\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}, <^{\mathbb{N}}\right),$$

i.e., the standard model of $\mathbb{N}$ with the natural ordering $<$. $\dashv$

Definition 2.3. An **assignment** in an S-structure $\mathfrak{A}$ is a mapping

$$\beta : \left\{v_i \mid i \in \mathbb{N}\right\} \to A. \qquad \dashv$$

Definition 2.4. An S-**interpretation** $\mathfrak{I}$ is a pair $(\mathfrak{A}, \beta)$ where $\mathfrak{A}$ is an S-structure and β is an assignment in $\mathfrak{A}$. $\dashv$

Definition 2.5. Let β be an assignment in $\mathfrak{A}$, $a \in A$, and x a variable. Then $\beta\frac{a}{x}$ is the assignment defined by

$$\beta\frac{a}{x}(y) := \begin{cases} a, & \text{if } y = x, \\ \beta(y), & \text{otherwise.} \end{cases}$$

Then, for the S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ we use $\mathfrak{I}\frac{a}{x}$ to denote the S-interpretation $\left(\mathfrak{A}, \beta\frac{a}{x}\right)$. $\dashv$

2.2 The satisfaction relation $\mathfrak{I} \models \varphi$

We fix an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$.

Definition 2.6. For every S-term t we define its **interpretation** $\mathfrak{I}(t)$ by induction on the construction of t.

(a) $\mathfrak{I}(x) = \beta(x)$ for a variable x.

(b) $\mathfrak{I}(c) = c^{\mathfrak{A}}$ for a constant $c \in S$.

(c) Let $f \in S$ be an n-ary function symbol and $t_1, \ldots, t_n$ S-terms. Then

$$\mathfrak{I}\big(ft_1 \cdots t_n\big) = f^{\mathfrak{A}}\big(\mathfrak{I}(t_1), \ldots, \mathfrak{I}(t_n)\big). \qquad \dashv$$

Example 2.7. Let $S := S_{Gr} = \{\circ, e\}$ and $\mathfrak{I} := (\mathfrak{A}, \beta)$ with $\mathfrak{A} = (\mathbb{R}, +, 0)$, $\beta(v_0) = 2$, and $\beta(v_2) = 6$. Then

$$\begin{aligned}\mathfrak{I}\big(v_0 \circ (e \circ v_2)\big) &= \mathfrak{I}(v_0) + \mathfrak{I}(e \circ v_2)\\ &= 2 + \big(\mathfrak{I}(e) + \mathfrak{I}(v_2)\big) = 2 + (0 + 6) = 2 + 6 = 8.\end{aligned} \qquad \dashv$$

Definition 2.8. Let φ be an S-formula. We define $\mathfrak{I} \models \varphi$ by induction on the construction of φ.

(a) $\mathfrak{I} \models t_1 \equiv t_2$ if $\mathfrak{I}(t_1) = \mathfrak{I}(t_2)$.

(b) $\mathfrak{I} \models Rt_1 \cdots t_n$ if $\big(\mathfrak{I}(t_1), \ldots, \mathfrak{I}(t_n)\big) \in R^{\mathfrak{A}}$.

(c) $\mathfrak{I} \models \neg\varphi$ if $\mathfrak{I} \not\models \varphi$ (i.e., it is **not** the case that $\mathfrak{I} \models \varphi$).

(d) $\mathfrak{I} \models (\varphi \wedge \psi)$ if $\mathfrak{I} \models \varphi$ and $\mathfrak{I} \models \psi$.

(e) $\mathfrak{I} \models (\varphi \vee \psi)$ if $\mathfrak{I} \models \varphi$ or $\mathfrak{I} \models \psi$.

(f) $\mathfrak{I} \models (\varphi \rightarrow \psi)$ if $\mathfrak{I} \models \varphi$ implies $\mathfrak{I} \models \psi$.

(g) $\mathfrak{I} \models (\varphi \leftrightarrow \psi)$ if $\big(\mathfrak{I} \models \varphi$ if and only if $\mathfrak{I} \models \psi\big)$.

(h) $\mathfrak{I} \models \forall x\varphi$ if for all $a \in A$ we have $\mathfrak{I}\frac{a}{x} \models \varphi$.

(i) $\mathfrak{I} \models \exists x\varphi$ if for some $a \in A$ we have $\mathfrak{I}\frac{a}{x} \models \varphi$.

If $\mathfrak{I} \models \varphi$, then $\mathfrak{I}$ is a **model** of φ, of $\mathfrak{I}$ **satisfies** φ.

Let Φ be a set of S-formulas. Then $\mathfrak{I} \models \Phi$ if $\mathfrak{I} \models \varphi$ for all $\varphi \in \Phi$. Similarly as above, we say that $\mathfrak{I}$ is a model of Φ, or $\mathfrak{I}$ satisfies Φ. $\dashv$

Example 2.9. Let $S := S_{Gr}$ and $\mathfrak{I} := (\mathfrak{A}, \beta)$ with $\mathfrak{A} = (\mathbb{R}, +, 0)$ and $\beta(x) = 9$ for all variables x. Then

$$\begin{aligned}\mathfrak{I} \models \forall v_0\, v_0 \circ e \equiv v_0 &\iff \text{for all } r \in \mathbb{R} \text{ we have } \mathfrak{I}\frac{r}{v_0} \models v_0 \circ e \equiv v_0,\\ &\iff \text{for all } r \in \mathbb{R} \text{ we have } r + 0 = r.\end{aligned} \qquad \dashv$$

Definition 2.10. Let Φ be a set of S-formulas and φ an S-formula. Then φ is a **consequence of** Φ, written $\Phi \models \varphi$, if for any interpretation $\mathfrak{I}$ it holds that $\mathfrak{I} \models \Phi$ implies $\mathfrak{I} \models \varphi$.

For simplicity, in case $\Phi = \{\psi\}$ we write $\psi \models \varphi$ instead of $\{\psi\} \models \varphi$. $\dashv$

Example 2.11. Let

$$\begin{aligned}\Phi_{Gr} := \big\{&\forall v_0 \forall v_1 \forall v_2\ (v_0 \circ v_1) \circ v_2 \equiv v_0 \circ (v_1 \circ v_2),\\ &\forall v_0\ v_0 \circ e \equiv v_0, \forall v_0 \exists v_1\ v_0 \circ v_1 \equiv e\big\}.\end{aligned}$$

Then it can be shown that

$$\Phi_{Gr} \models \forall v_0\ e \circ v_0 \equiv v_0.$$

and

$$\Phi_{Gr} \models \forall v_0 \exists v_1\ v_1 \circ v_0 \equiv e. \qquad \dashv$$

Definition 2.12. An S-formula φ is **valid**, written $\models \varphi$, if $\emptyset \models \varphi$. Or equivalently, $\mathfrak{I} \models \varphi$ for any $\mathfrak{I}$. $\dashv$

Definition 2.13. An S-formula φ is **satisfiable**, if there exists an S-interpretation $\mathfrak{I}$ with $\mathfrak{I} \models \varphi$. A set Φ of S-formulas is satisfiable if there exists an S-interpretation $\mathfrak{I}$ such that $\mathfrak{I} \models \varphi$ for every $\varphi \in \Phi$. ⊣

The next lemma is essentially the method of **proof by contradiction**.

Lemma 2.14. *Let Φ be a set of S-formulas and φ an S-formula. Then $\Phi \models \varphi$ if and only if $\Phi \cup \{\neg\varphi\}$ is not satisfiable.* ⊣

Proof:

$$\begin{aligned}
\Phi \models \varphi &\iff \text{Every model of } \Phi \text{ is a model of } \varphi, \\
&\iff \text{there is no model } \mathfrak{I} \text{ with } \mathfrak{I} \models \Phi \text{ and } \mathfrak{I} \not\models \varphi, \\
&\iff \text{there is no model } \mathfrak{I} \text{ with } \mathfrak{I} \models \Phi \cup \{\neg\varphi\}, \\
&\iff \Phi \cup \{\neg\varphi\} \text{ is not satisfiable.}
\end{aligned}$$

□

Definition 2.15. Two S-formulas φ and ψ are **logic equivalent** if $\varphi \models \psi$ and $\psi \models \varphi$. ⊣

Example 2.16. Let φ be an S-formula. We define a logic equivalent φ^* which does not contain the logic symbols $\wedge, \rightarrow, \leftrightarrow, \forall$.

$$\begin{aligned}
\varphi^* &:= \varphi \qquad \text{if } \varphi \text{ is atomic,} \\
(\neg\varphi)^* &:= \neg\varphi^*, \\
(\varphi \wedge \psi)^* &:= \neg(\neg\varphi^* \vee \neg\psi^*), \\
(\varphi \vee \psi)^* &:= (\varphi^* \vee \psi^*), \\
(\varphi \rightarrow \psi)^* &:= (\neg\varphi^* \vee \psi^*), \\
(\varphi \leftrightarrow \psi)^* &:= \neg(\varphi^* \vee \psi^*) \vee \neg(\neg\varphi^* \vee \neg\psi^*), \\
(\forall x\varphi)^* &:= \neg\exists x\neg\varphi^*, \\
(\exists x\varphi)^* &:= \exists x\varphi^*.
\end{aligned}$$

Thus, it suffices to consider $\neg, \vee, \exists$ as the only logic symbols in any given φ. ⊣