Mathematical Logic (III)

Yijia Chen

1 The Semantics of First-order Logic

1.1 Structures and interpretations

We fix a symbol set S.

Definition 1.1. An S**-structure** is a pair $\mathfrak{A} = (A, \mathfrak{a})$ which satisfies the following conditions.

1. $A \neq \emptyset$ is the **universe** of $\mathfrak{A}$.
2. $\mathfrak{a}$ is a function defined on S such that:
   (a) Let $R \in S$ be an n-ary relation symbol. Then $\mathfrak{a}(R) \subseteq A^n$.
   (b) Let $f \in S$ be an n-ary function symbol. Then $\mathfrak{a}(f) : A^n \to A$.
   (c) $\mathfrak{a}(c) \in A$ for every constant $c \in S$.

For better readability, we write $R^{\mathfrak{A}}$, $f^{\mathfrak{A}}$, and $c^{\mathfrak{A}}$, or even R^A, f^A, and c^A, instead of $\mathfrak{a}(R)$, $\mathfrak{a}(f)$, and $\mathfrak{a}(c)$. Thus for $S = \{R, f, c\}$ we might write an S-structure as

$$\mathfrak{A} = \left(A, R^{\mathfrak{A}}, f^{\mathfrak{A}}, c^{\mathfrak{A}}\right) = \left(A, R^A, f^A, c^A\right). \qquad \dashv$$

Examples 1.2.

1. For $S_{\mathrm{Ar}} := \{+, \cdot, 0, 1\}$ the S_{Ar}-structure

$$\mathfrak{N} = \left(\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}\right)$$

   is the standard model of natural numbers with addition, multiplication, and constants 0 and 1.

2. For $S_{\mathrm{Ar}}^{<} := \{+, \cdot, 0, 1, <\}$ we have an $S_{\mathrm{Ar}}^{<}$-structure

$$\mathfrak{N}^{<} = \left(\mathbb{N}, +^{\mathbb{N}}, \cdot^{\mathbb{N}}, 0^{\mathbb{N}}, 1^{\mathbb{N}}, <^{\mathbb{N}}\right),$$

   i.e., the standard model of $\mathbb{N}$ with the natural ordering $<$. $\dashv$

Definition 1.3. An **assignment** in an S-structure $\mathfrak{A}$ is a mapping

$$\beta : \left\{v_i \mid i \in \mathbb{N}\right\} \to A. \qquad \dashv$$

Definition 1.4. An S**-interpretation** $\mathfrak{I}$ is a pair $(\mathfrak{A}, \beta)$ where $\mathfrak{A}$ is an S-structure and β is an assignment in $\mathfrak{A}$. $\dashv$

Definition 1.5. Let β be an assignment in $\mathfrak{A}$, $a \in A$, and x a variable. Then $\beta\frac{a}{x}$ is the assignment defined by

$$\beta\frac{a}{x}(y) := \begin{cases} a, & \text{if } y = x, \\ \beta(y), & \text{otherwise.} \end{cases}$$

Then, for the S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ we use $\mathfrak{I}\frac{a}{x}$ to denote the S-interpretation $\left(\mathfrak{A}, \beta\frac{a}{x}\right)$. $\dashv$

1.2 The satisfaction relation $\mathfrak{I} \models \varphi$

We fix an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$.

Definition 1.6. For every S-term t we define its **interpretation** $\mathfrak{I}(t)$ by induction on the construction of t.

(a) $\mathfrak{I}(x) = \beta(x)$ for a variable x.

(b) $\mathfrak{I}(c) = c^{\mathfrak{A}}$ for a constant $c \in S$.

(c) Let $f \in S$ be an n-ary function symbol and $t_1, \dots, t_n$ S-terms. Then

$$\mathfrak{I}\big(ft_1 \cdots t_n\big) = f^{\mathfrak{A}}\big(\mathfrak{I}(t_1), \dots, \mathfrak{I}(t_n)\big). \qquad \dashv$$

Example 1.7. Let $S := S_{\text{Gr}} = \{\circ, e\}$ and $\mathfrak{I} := (\mathfrak{A}, \beta)$ with $\mathfrak{A} = (\mathbb{R}, +, 0)$, $\beta(v_0) = 2$, and $\beta(v_2) = 6$. Then

$$\begin{aligned} \mathfrak{I}\big(v_0 \circ (e \circ v_2)\big) &= \mathfrak{I}(v_0) + \mathfrak{I}(e \circ v_2) \\ &= 2 + \big(\mathfrak{I}(e) + \mathfrak{I}(v_2)\big) = 2 + (0 + 6) = 2 + 6 = 8. \end{aligned} \qquad \dashv$$

Definition 1.8. Let φ be an S-formula. We define $\mathfrak{I} \models \varphi$ by induction on the construction of φ.

(a) $\mathfrak{I} \models t_1 \equiv t_2$ if $\mathfrak{I}(t_1) = \mathfrak{I}(t_2)$.

(b) $\mathfrak{I} \models Rt_1 \cdots t_n$ if $\big(\mathfrak{I}(t_1), \dots, \mathfrak{I}(t_n)\big) \in R^{\mathfrak{A}}$.

(c) $\mathfrak{I} \models \neg\varphi$ if $\mathfrak{I} \not\models \varphi$ (i.e., it is **not** the case that $\mathfrak{I} \models \varphi$).

(d) $\mathfrak{I} \models (\varphi \wedge \psi)$ if $\mathfrak{I} \models \varphi$ and $\mathfrak{I} \models \psi$.

(e) $\mathfrak{I} \models (\varphi \vee \psi)$ if $\mathfrak{I} \models \varphi$ or $\mathfrak{I} \models \psi$.

(f) $\mathfrak{I} \models (\varphi \to \psi)$ if $\mathfrak{I} \models \varphi$ implies $\mathfrak{I} \models \psi$.

(g) $\mathfrak{I} \models (\varphi \leftrightarrow \psi)$ if $\big(\mathfrak{I} \models \varphi$ if and only if $\mathfrak{I} \models \psi\big)$.

(h) $\mathfrak{I} \models \forall x\varphi$ if for all $a \in A$ we have $\mathfrak{I}\frac{a}{x} \models \varphi$.

(i) $\mathfrak{I} \models \exists x\varphi$ if for some $a \in A$ we have $\mathfrak{I}\frac{a}{x} \models \varphi$.

If $\mathfrak{I} \models \varphi$, then $\mathfrak{I}$ is a **model** of φ, of $\mathfrak{I}$ **satisfies** φ.

Let Φ be a set of S-formulas. Then $\mathfrak{I} \models \Phi$ if $\mathfrak{I} \models \varphi$ for all $\varphi \in \Phi$. Similarly as above, we say that $\mathfrak{I}$ is a model of Φ, or $\mathfrak{I}$ satisfies Φ. $\dashv$

Example 1.9. Let $S := S_{\text{Gr}}$ and $\mathfrak{I} := (\mathfrak{A}, \beta)$ with $\mathfrak{A} = (\mathbb{R}, +, 0)$ and $\beta(x) = 9$ for all variables x. Then

$$\begin{aligned} \mathfrak{I} \models \forall v_0\, v_0 \circ e \equiv v_0 &\iff \text{for all } r \in \mathbb{R} \text{ we have } \mathfrak{I}\frac{r}{v_0} \models v_0 \circ e \equiv v_0, \\ &\iff \text{for all } r \in \mathbb{R} \text{ we have } r + 0 = r. \end{aligned} \qquad \dashv$$

Definition 1.10. Let Φ be a set of S-formulas and φ an S-formula. Then φ is a **consequence of** Φ, written $\Phi \models \varphi$, if for any interpretation $\mathfrak{I}$ it holds that $\mathfrak{I} \models \Phi$ implies $\mathfrak{I} \models \varphi$.

For simplicity, in case $\Phi = \{\psi\}$ we write $\psi \models \varphi$ instead of $\{\psi\} \models \varphi$. $\dashv$

Example 1.11. Let

$$\Phi_{\mathrm{Gr}} := \{\forall v_0 \forall v_1 \forall v_2\ (v_0 \circ v_1) \circ v_2 \equiv v_0 \circ (v_1 \circ v_2),\\ \forall v_0\ v_0 \circ e \equiv v_0, \forall v_0 \exists v_1\ v_0 \circ v_1 \equiv e\}.$$

Then it can be shown that

$$\Phi_{\mathrm{Gr}} \models \forall v_0\ e \circ v_0 \equiv v_0.$$

and

$$\Phi_{\mathrm{Gr}} \models \forall v_0 \exists v_1\ v_1 \circ v_0 \equiv e.$$

⊣

Definition 1.12. An S-formula φ is **valid**, written $\models \varphi$, if $\emptyset \models \varphi$. Or equivalently, $\mathfrak{I} \models \varphi$ for any $\mathfrak{I}$. ⊣

Definition 1.13. An S-formula φ is **satisfiable**, if there exists an S-interpretation $\mathfrak{I}$ with $\mathfrak{I} \models \varphi$. A set Φ of S-formulas is satisfiable if there exists an S-interpretation $\mathfrak{I}$ such that $\mathfrak{I} \models \varphi$ for every $\varphi \in \Phi$. ⊣

The next lemma is essentially the method of **proof by contradiction**.

Lemma 1.14. *Let Φ be a set of S-formulas and φ an S-formula. Then $\Phi \models \varphi$ if and only if $\Phi \cup \{\neg\varphi\}$ is not satisfiable.* ⊣

Proof:

$$\begin{aligned} \Phi \models \varphi &\iff \text{Every model of } \Phi \text{ is a model of } \varphi, \\ &\iff \text{there is no model } \mathfrak{I} \text{ with } \mathfrak{I} \models \Phi \text{ and } \mathfrak{I} \not\models \varphi, \\ &\iff \text{there is no model } \mathfrak{I} \text{ with } \mathfrak{I} \models \Phi \cup \{\neg\varphi\}, \\ &\iff \Phi \cup \{\neg\varphi\} \text{ is not satisfiable.} \end{aligned}$$

□

Definition 1.15. Two S-formulas φ and ψ are **logic equivalent** if $\varphi \models \psi$ and $\psi \models \varphi$. ⊣

Example 1.16. Let φ be an S-formula. We define a logic equivalent φ^* which does not contain the logic symbols $\wedge, \rightarrow, \leftrightarrow, \forall$.

$$\begin{aligned} \varphi^* &:= \varphi \qquad \text{if } \varphi \text{ is atomic,} \\ (\neg\varphi)^* &:= \neg\varphi^*, \\ (\varphi \wedge \psi)^* &:= \neg(\neg\varphi^* \vee \neg\psi^*), \\ (\varphi \vee \psi)^* &:= (\varphi^* \vee \psi^*), \\ (\varphi \rightarrow \psi)^* &:= (\neg\varphi^* \vee \psi^*), \\ (\varphi \leftrightarrow \psi)^* &:= \neg(\varphi^* \vee \psi^*) \vee \neg(\neg\varphi^* \vee \neg\psi^*), \\ (\forall x \varphi)^* &:= \neg\exists x \neg\varphi^*, \\ (\exists x \varphi)^* &:= \exists x \varphi^*. \end{aligned}$$

Thus, it suffices to consider $\neg, \vee, \exists$ as the only logic symbols in any given φ. ⊣

Lemma 1.17 (The Coincidence Lemma). *For $i \in \{1, 2\}$ let $\mathfrak{I}_i = (\mathfrak{A}_i, \beta_i)$ be an S_i-interpretation such that $A_1 = A_2$ and every symbol in $S := S_1 \cap S_2$ has the same interpretation in $\mathfrak{A}_1$ and $\mathfrak{A}_2$.*

(a) Let t be an S-term (thus also an S_1-term and an S_2-term). Assume further that $\beta_1(x) = \beta_2(x)$ for every variable $x \in \mathrm{var}(t)$. Then $\mathfrak{I}_1(t) = \mathfrak{I}_2(t)$.

(b) Let φ be an S-formula where $\beta_1(x) = \beta_2(x)$ for every $x \in \mathrm{free}(\varphi)$. Then

$$\mathfrak{I}_1 \models \varphi \iff \mathfrak{I}_2 \models \varphi.$$

⊣

Proof: (a) We prove by induction on t.

- $t = x$. Then $\mathfrak{I}_1(x) = \beta_1(x) = \beta_2(x) = \mathfrak{I}_2(x)$.
- $t = c$. We deduce $\mathfrak{I}_1(c) = c^{\mathfrak{A}_1} = c^{\mathfrak{A}_2} = \mathfrak{I}_2(x)$.
- $t = ft_1 \cdots t_n$. It holds that

$$\begin{aligned}\mathfrak{I}_1(ft_1 \cdots t_n) &= f^{\mathfrak{A}_1}\big(\mathfrak{I}_1(t_1), \ldots, \mathfrak{I}_2(t_n)\big)\\ &= f^{\mathfrak{A}_2}\big(\mathfrak{I}_1(t_1), \ldots, \mathfrak{I}_1(t_n)\big)\\ &= f^{\mathfrak{A}_2}\big(\mathfrak{I}_2(t_1), \ldots, \mathfrak{I}_2(t_n)\big)\\ &= \mathfrak{I}_2(ft_1 \cdots t_n).\end{aligned}$$

(b) The induction proof is on the structure of φ.

- $\varphi = t_1 \equiv t_2$. We have

$$\begin{aligned}\mathfrak{I}_1 \models t_1 \equiv t_2 &\iff \mathfrak{I}_1(t_1) = \mathfrak{I}_1(t_2)\\ &\iff \mathfrak{I}_2(t_1) = \mathfrak{I}_2(t_2) && \text{(by (a))}\\ &\iff \mathfrak{I}_2 \models t_1 \equiv t_2.\end{aligned}$$

- $\varphi = Rt_1 \cdots t_n$. Then

$$\begin{aligned}\mathfrak{I}_1 \models Rt_1 \cdots t_n &\iff \big(\mathfrak{I}_1(t_1), \ldots, \mathfrak{I}_1(t_n)\big) \in R^{\mathfrak{A}_1}\\ &\iff \big(\mathfrak{I}_1(t_1), \ldots, \mathfrak{I}_1(t_n)\big) \in R^{\mathfrak{A}_2}\\ &\iff \big(\mathfrak{I}_2(t_1), \ldots, \mathfrak{I}_2(t_n)\big) \in R^{\mathfrak{A}_2}\\ &\iff \mathfrak{I}_2 \models Rt_1 \cdots t_n.\end{aligned}$$

- $\varphi = \neg\psi$. We conclude

$$\mathfrak{I}_1 \models \neg\psi \iff \mathfrak{I}_1 \not\models \psi \iff \mathfrak{I}_2 \not\models \psi \iff \mathfrak{I}_2 \models \neg\psi.$$

- $\varphi = (\psi \vee \chi)$.

$$\begin{aligned}\mathfrak{I}_1 \models (\psi \vee \chi) &\iff \mathfrak{I}_1 \models \psi \text{ or } \mathfrak{I}_1 \models \chi\\ &\iff \mathfrak{I}_2 \models \psi \text{ or } \mathfrak{I}_2 \models \chi\\ &\iff \mathfrak{I}_2 \models (\psi \vee \chi).\end{aligned}$$

- $\varphi = \exists x\psi$.

$$\begin{aligned}\mathfrak{I}_1 \models \exists x\psi &\iff \text{for some } a \in A_1 \text{ we have } \mathfrak{I}_1\frac{a}{x} \models \psi\\ &\iff \text{for some } a \in A_1 \text{ we have } \mathfrak{I}_2\frac{a}{x} \models \psi\\ &\qquad\qquad \Big(\text{by induction hypothesis on } \mathfrak{I}_1\frac{a}{x}, \mathfrak{I}_2\frac{a}{x}, \text{ and } \psi\Big)\\ &\iff \mathfrak{I}_2 \models \exists x\psi.\end{aligned}$$

□

Remark 1.18. Let $\varphi \in L^S_n$, i.e., φ is an S-formula with free$(\varphi) \subseteq \{v_0, \ldots, v_{n-1}\}$. By the coincidence lemma whether $\mathfrak{I} = (\mathfrak{A}, \beta) \models \varphi$ is completely determined by $\mathfrak{A}$ and $\beta(v_0), \ldots, \beta(v_{n-1})$. So in case $\mathfrak{I} \models \varphi$ we can write

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_{n-1}]$$

where $a_i := \beta(v_i)$ for $0 \leqslant i < n$. In particular, if φ is an S-sentence, i.e., $\varphi \in L^S_0$, then $\mathfrak{A} \models \varphi$ is well-defined.

Similarly, we write

$$t^{\mathfrak{A}}[a_0, \ldots, a_{n-1}]$$

instead of $\mathfrak{I}(t)$. ⊣

Definition 1.19. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two S-structures.

(a) A mapping $\pi : A \to B$ is an **isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$** (in short $\pi : \mathfrak{A} \cong \mathfrak{B}$) if the following conditions are satisfied.

(i) π is a bijection.

(ii) For any n-ary relation symbol $R \in S$ and $a_0, \ldots, a_{n-1} \in A$

$$(a_0, \ldots, a_{n-1}) \in R^{\mathfrak{A}} \iff \big(\pi(a_0), \ldots, \pi(a_{n-1})\big) \in R^{\mathfrak{B}}.$$

(iii) For any n-ary function symbol $f \in S$ and $a_0, \ldots, a_{n-1} \in A$

$$\pi(f^{\mathfrak{A}}(a_0, \ldots, a_{n-1})) = f^{\mathfrak{B}}(\pi(a_0), \ldots, \pi(a_{n-1})).$$

(iv) For any constant $c \in S$

$$\pi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

(b) $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, written $\mathfrak{A} \cong \mathfrak{B}$, if there is an isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$. ⊣

Observe that the above definition is not symmetric. However we can easily show:

Lemma 1.20. *$\cong$ is an equivalence relation. That is, for all S-structures $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$*

1. *$\mathfrak{A} \cong \mathfrak{A}$;*
2. *$\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{B} \cong \mathfrak{A}$;*
3. *if $\mathfrak{A} \cong \mathfrak{B}$ and $\mathfrak{B} \cong \mathfrak{C}$, then $\mathfrak{A} \cong \mathfrak{C}$.* ⊣

Lemma 1.21 (The Isomorphism Lemma). *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two isomorphic S-structures. Then for every S-sentence φ*

$$\mathfrak{A} \models \varphi \iff \mathfrak{B} \models \varphi.$$

⊣

Proof: Let β be an assignment in $\mathfrak{A}$. By the coincidence lemma, it suffices to show that there is an assignment β' in $\mathfrak{B}$ such that

$$(\mathfrak{A}, \beta) \models \varphi \iff (\mathfrak{B}, \beta') \models \varphi, \tag{1}$$

where φ is an S-sentence.

Let $\pi : \mathfrak{A} \cong \mathfrak{B}$ and we define an assignment β^π in $\mathfrak{B}$ by

$$\beta^\pi(x) := \pi(\beta(x))$$

for any variable x. Then we prove for any S-**formula** φ

$$(\mathfrak{A}, \beta) \models \varphi \iff (\mathfrak{B}, \beta^\pi) \models \varphi, \tag{2}$$

which certainly generalizes (1). To simplify notation, let $\mathfrak{I} := (\mathfrak{A}, \beta)$ and $\mathfrak{I}^\pi := (\mathfrak{B}, \beta^\pi)$. First, it is routine to verify that for every S-term t

$$\pi(\mathfrak{I}(t)) = \mathfrak{I}^\pi(t). \tag{3}$$

Then we prove (2) by induction on the construction of S-formula φ.

- $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned}
\mathfrak{I} \models t_1 \equiv t_2 &\iff \mathfrak{I}(t_1) = \mathfrak{I}(t_2) \\
&\iff \pi(\mathfrak{I}(t_1)) = \pi(\mathfrak{I}(t_2)) && \text{(since } \pi \text{ is an injection)} \\
&\iff \mathfrak{I}^\pi(t_1) = \mathfrak{I}^\pi(t_2) && \text{(by (3))} \\
&\iff \mathfrak{I}^\pi \models t_1 \equiv t_2.
\end{aligned}$$

- $\varphi = Rt_1 \cdots t_n$.

$$\begin{aligned}
\mathfrak{I} \models Rt_1 \cdots t_n &\iff \big(\mathfrak{I}(t_1), \ldots, \mathfrak{I}(t_n)\big) \in R^{\mathfrak{A}} \\
&\iff \big(\pi(\mathfrak{I}(t_1)), \ldots, \pi(\mathfrak{I}(t_n))\big) \in R^{\mathfrak{B}} \\
&\iff \big(\mathfrak{I}^\pi(t_1), \ldots, \mathfrak{I}^\pi(t_n)\big) \in R^{\mathfrak{B}} && \text{(by (3))} \\
&\iff \mathfrak{I}^\pi \models Rt_1 \cdots t_n.
\end{aligned}$$

- $\varphi = \neg\psi$. It follows that $\mathfrak{I} \models \neg\psi \iff \mathfrak{I} \not\models \psi \iff \mathfrak{I}^\pi \not\models \iff \mathfrak{I}^\pi \models \neg\psi$.
- $\varphi = \psi \vee \chi$. The inductive argument is similar to the above $\neg\psi$.
- $\varphi = \exists x\psi$. This is again the most complicated case.

$$\begin{aligned}
\mathfrak{I} \models \exists x\psi &\iff \text{there exists an } a \in A \text{ such that } \mathfrak{I}\frac{a}{x} = \left(\mathfrak{A}, \beta\frac{a}{x}\right) \models \psi \\
&\iff \text{there exists an } a \in A \text{ such that } \left(\mathfrak{I}\frac{a}{x}\right)^\pi = \left(\mathfrak{A}, \beta\frac{a}{x}\right)^\pi \models \psi, \\
&\qquad \left(\text{by induction hypothesis on } \mathfrak{I}\frac{a}{x}, \left(\mathfrak{I}\frac{a}{x}\right)^\pi, \text{ and } \psi\right) \\
&\qquad \text{that is, there exists an } a \in A \text{ such that } \left(\mathfrak{B}, \beta^\pi\frac{\pi(a)}{x}\right) \models \psi \\
&\iff \text{there exists a } b \in B \text{ such that } \left(\mathfrak{B}, \beta^\pi\frac{b}{x}\right) \models \psi \qquad \text{(since } \pi \text{ is surjective)} \\
&\qquad \text{i.e., there exists a } b \in B \text{ with } \mathfrak{I}^\pi\frac{b}{x} = (\mathfrak{B}, \beta^\pi)\frac{b}{x} \models \psi \\
&\iff \mathfrak{I}^\pi \models \exists x\psi.
\end{aligned}$$

This finishes the proof. □

Corollary 1.22. *Let* $\pi : \mathfrak{A} \cong \mathfrak{B}$ *and* $\varphi \in L_n^S$. *Then for every* $a_0, \ldots, a_{n-1}$

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_{n-1}] \iff \mathfrak{B} \models \varphi\big[\pi(a_0), \ldots, \pi(a_{n-1})\big]$$

⊣