Mathematical Logic (IV)

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1 The Semantics of First-order Logic

1.1 Isomorphisms

Definition 1.1. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two S-structures.

(a) A mapping $\pi : A \to B$ is an **isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$** (in short $\pi : \mathfrak{A} \cong \mathfrak{B}$) if the following conditions are satisfied.

(i) π is a bijection.

(ii) For any n-ary relation symbol $R \in S$ and $a_0, \dots, a_{n-1} \in A$

$$(a_0, \dots, a_{n-1}) \in R^{\mathfrak{A}} \iff \big(\pi(a_0), \dots, \pi(a_{n-1})\big) \in R^{\mathfrak{B}}.$$

(iii) For any n-ary function symbol $f \in S$ and $a_0, \dots, a_{n-1} \in A$

$$\pi(f^{\mathfrak{A}}(a_0, \dots, a_{n-1})) = f^{\mathfrak{B}}(\pi(a_0), \dots, \pi(a_{n-1})).$$

(iv) For any constant $c \in S$

$$\pi(c^{\mathfrak{A}}) = c^{\mathfrak{B}}.$$

(b) $\mathfrak{A}$ and $\mathfrak{B}$ are isomorphic, written $\mathfrak{A} \cong \mathfrak{B}$, if there is an isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$. $\dashv$

Observe that the above definition is not symmetric. However we can easily show:

Lemma 1.2. *$\cong$ is an equivalence relation. That is, for all S-structures $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$*

1. *$\mathfrak{A} \cong \mathfrak{A}$;*
2. *$\mathfrak{A} \cong \mathfrak{B}$ implies $\mathfrak{B} \cong \mathfrak{A}$;*
3. *if $\mathfrak{A} \cong \mathfrak{B}$ and $\mathfrak{B} \cong \mathfrak{C}$, then $\mathfrak{A} \cong \mathfrak{C}$.* $\dashv$

Lemma 1.3 (The Isomorphism Lemma)**.** *Let $\mathfrak{A}$ and $\mathfrak{B}$ be two isomorphic S-structures. Then for every S-sentence φ*

$$\mathfrak{A} \models \varphi \iff \mathfrak{B} \models \varphi.$$

$\dashv$

Proof: Let β be an assignment in $\mathfrak{A}$. By the coincidence lemma, it suffices to show that there is an assignment β' in $\mathfrak{B}$ such that

$$(\mathfrak{A}, \beta) \models \varphi \iff (\mathfrak{B}, \beta') \models \varphi, \tag{1}$$

where φ is an S-sentence.

Let $\pi : \mathfrak{A} \cong \mathfrak{B}$ and we define an assignment β^π in $\mathfrak{B}$ by

$$\beta^\pi(x) := \pi(\beta(x))$$

for any variable x. Then we prove for any S-**formula** φ

$$(\mathfrak{A}, \beta) \models \varphi \quad \Longleftrightarrow \quad (\mathfrak{B}, \beta^\pi) \models \varphi, \tag{2}$$

which certainly generalizes (1). To simplify notation, let $\mathfrak{I} := (\mathfrak{A}, \beta)$ and $\mathfrak{I}^\pi := \big(\mathfrak{B}, \beta^\pi\big)$. First, it is routine to verify that for every S-term t

$$\pi(\mathfrak{I}(t)) = \mathfrak{I}^\pi(t). \tag{3}$$

Then we prove (2) by induction on the construction of S-formula φ.

- $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned}
\mathfrak{I} \models t_1 \equiv t_2 &\Longleftrightarrow \mathfrak{I}(t_1) = \mathfrak{I}(t_2) \\
&\Longleftrightarrow \pi(\mathfrak{I}(t_1)) = \pi(\mathfrak{I}(t_2)) && \text{(since } \pi \text{ is an injection)} \\
&\Longleftrightarrow \mathfrak{I}^\pi(t_1) = \mathfrak{I}^\pi(t_2) && \text{(by (3))} \\
&\Longleftrightarrow \mathfrak{I}^\pi \models t_1 \equiv t_2.
\end{aligned}$$

- $\varphi = Rt_1 \cdots t_n$.

$$\begin{aligned}
\mathfrak{I} \models Rt_1 \cdots t_n &\Longleftrightarrow \big(\mathfrak{I}(t_1), \ldots, \mathfrak{I}(t_n)\big) \in R^{\mathfrak{A}} \\
&\Longleftrightarrow \big(\pi(\mathfrak{I}(t_1)), \ldots, \pi(\mathfrak{I}(t_n))\big) \in R^{\mathfrak{B}} \\
&\Longleftrightarrow \big(\mathfrak{I}^\pi(t_1), \ldots, \mathfrak{I}^\pi(t_n)\big) \in R^{\mathfrak{B}} && \text{(by (3))} \\
&\Longleftrightarrow \mathfrak{I}^\pi \models Rt_1 \cdots t_n.
\end{aligned}$$

- $\varphi = \neg\psi$. It follows that $\mathfrak{I} \models \neg\psi \iff \mathfrak{I} \not\models \psi \iff \mathfrak{I}^\pi \not\models \iff \mathfrak{I}^\pi \models \neg\psi$.
- $\varphi = \psi \vee \chi$. The inductive argument is similar to the above $\neg\psi$.
- $\varphi = \exists x\psi$. This is again the most complicated case.

$$\begin{aligned}
\mathfrak{I} \models \exists x\psi &\Longleftrightarrow \text{there exists an } a \in A \text{ such that } \mathfrak{I}\frac{a}{x} = \left(\mathfrak{A}, \beta\frac{a}{x}\right) \models \psi \\
&\Longleftrightarrow \text{there exists an } a \in A \text{ such that } \left(\mathfrak{I}\frac{a}{x}\right)^\pi = \left(\mathfrak{A}, \beta\frac{a}{x}\right)^\pi \models \psi, \\
&\qquad \left(\text{by induction hypothesis on } \mathfrak{I}\frac{a}{x}, \left(\mathfrak{I}\frac{a}{x}\right)^\pi, \text{ and } \psi\right) \\
&\qquad \text{that is, there exists an } a \in A \text{ such that } \left(\mathfrak{B}, \beta^\pi\frac{\pi(a)}{x}\right) \models \psi \\
&\Longleftrightarrow \text{there exists a } b \in B \text{ such that } \left(\mathfrak{B}, \beta^\pi\frac{b}{x}\right) \models \psi \qquad \text{(since } \pi \text{ is surjective)} \\
&\qquad \text{i.e., there exists a } b \in B \text{ with } \mathfrak{I}^\pi\frac{b}{x} = \big(\mathfrak{B}, \beta^\pi\big)\frac{b}{x} \models \psi \\
&\Longleftrightarrow \mathfrak{I}^\pi \models \exists x\psi.
\end{aligned}$$

This finishes the proof. □

Corollary 1.4. *Let* $\pi : \mathfrak{A} \cong \mathfrak{B}$ *and* $\varphi \in L^S_n$. *Then for every* $a_0, \ldots, a_{n-1}$

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_{n-1}] \quad \Longleftrightarrow \quad \mathfrak{B} \models \varphi\big[\pi(a_0), \ldots, \pi(a_{n-1})\big]$$

⊣

1.2 Substitution

In mathematics, when writing $f(y+10)$ we plug the value of $y+10$ into $f(x)$. We will do the same for $\varphi(x)$ where we want to substitute x by a term t. This is not completely trivial, e.g.,

$$\varphi(x) = \exists z\, z + z \equiv x \quad \text{and} \quad t = x + z.$$

It is obviously wrong for

$$\exists z\, z + z \equiv x + z.$$

Definition 1.5. Let t be an S-term, $x_0, \dots, x_r$ variables, and $t_0, \dots, t_r$ S-terms. Then the term

$$t\frac{t_0, \dots, t_r}{x_0, \dots, x_r}$$

is defined inductively as follows.

(a) Let $t = x$ be a variable. Then

$$t\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \begin{cases} t_i & \text{if } x = x_i \text{ for some } 0 \leqslant i \leqslant r \\ x & \text{otherwise.} \end{cases}$$

(b) For a constant $t = c$

$$c\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := c.$$

(c) For a function term

$$ft'_1 \dots t'_n \frac{t_0, \dots, t_r}{x_0, \dots, x_r} := ft'_1 \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \dots t'_n \frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

⊣

Definition 1.6. Let φ be an S-formula, $x_0, \dots, x_r$ variables, and $t_0, \dots, t_r$ S-terms. We define

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r}$$

inductively as follow.

(a) Assume $\varphi = t'_1 \equiv t'_2$. Then

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := t'_1\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \equiv t'_2\frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(b) Let $\varphi = Rt'_1 \dots t'_n$. We set

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := Rt'_1\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \dots t'_n\frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(c) For $\varphi = \neg\psi$

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \neg\psi\frac{t_0, \dots, t_r}{x_0, \dots, x_r}.$$

(d) For $\varphi = (\psi_1 \vee \psi_2)$

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \left(\psi_1\frac{t_0, \dots, t_r}{x_0, \dots, x_r} \vee \psi_2\frac{t_0, \dots, t_r}{x_0, \dots, x_r}\right).$$

(e) Assume $\varphi = \exists x\psi$. Let $x_{i_1}, \dots, x_{i_s}$ ($i_1 < \dots < i_s$) be the variables x_i in $x_0, \dots, x_r$ with $x_i \in \text{free}(\exists x\psi)$ and $x_i \neq t_i$. In particular, $x \neq x_{i_1}, \dots, x \neq x_{i_s}$. Then

$$\varphi\frac{t_0, \dots, t_r}{x_0, \dots, x_r} := \exists u \left[\psi\frac{t_{i_1}, \dots, t_{i_s}, u}{x_{i_1}, \dots, x_{i_s}, x}\right],$$

where $u = x$ if x does not occur in $t_{i_1}, \dots, t_{t_s}$; otherwise u is the first variable in $\{v_0, v_1, v_2, \dots\}$ which does not occur in $\psi, t_{i_1}, \dots, t_{i_s}$. ⊣

Examples 1.7. 1.

$$[Pv_0fv_1v_2]\frac{v_2, v_0, v_1}{v_1, v_2, v_3} = Pv_0fv_2v_0.$$

2.

$$[\exists v_0\ Pv_0fv_1v_2]\frac{v_4, fv_1v_1}{v_0, v_2} = \exists v_0 \left[Pv_0fv_1v_2\frac{fv_1v_1, v_0}{v_2, v_0}\right] = \exists v_0\ Pv_0fv_1fv_1v_1.$$

3.

$$[\exists v_0\ Pv_0fv_1v_2]\frac{v_0, v_2, v_4}{v_1, v_2, v_0} = \exists v_3 \left[Pv_0fv_1v_2\frac{v_0, v_3}{v_1, v_0}\right] = \exists v_3\ Pv_3fv_0v_2.$$ ⊣

Definition 1.8. Let β be an assignment in $\mathfrak{A}$ and $a_0, \dots, a_r \in A$. Then

$$\beta\frac{a_0, \dots, a_r}{x_0, \dots, x_r}$$

is an assignment in $\mathfrak{A}$ defined by

$$\beta\frac{a_0, \dots, a_r}{x_0, \dots, x_r}(y) := \begin{cases} a_i & \text{if } y = x_i \text{ for } 0 \leqslant i \leqslant r \\ \beta(y) & \text{otherwise.} \end{cases}$$

For an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ we let

$$\mathfrak{I}\frac{a_0, \dots, a_r}{x_0, \dots, x_r} := \left(\mathfrak{A}, \beta\frac{a_0, \dots, a_r}{x_0, \dots, x_r}\right).$$ ⊣

Lemma 1.9 (The Substitution Lemma). *(a) For every S-term* t

$$\mathfrak{I}\left(t\frac{t_0, \dots, t_r}{x_0, \dots x_r}\right) = \mathfrak{I}\frac{\mathfrak{I}(t_0), \dots, \mathfrak{I}(t_r)}{x_0, \dots x_r}(t).$$

(b) For every S-formula φ

$$\mathfrak{I} \models \varphi\frac{t_0, \dots, t_r}{x_0, \dots x_r} \iff \mathfrak{I}\frac{\mathfrak{I}(t_0), \dots, \mathfrak{I}(t_r)}{x_0, \dots x_r} \models \varphi.$$ ⊣

Proof: (a) Assume $t = x$. If $x \neq x_i$ for all $0 \leqslant i \leqslant r$, then

$$t\frac{t_0, \dots, t_r}{x_0, \dots, x_r} = x.$$

Therefore,

$$\mathfrak{I}\left(t\frac{t_0, \dots, t_r}{x_0, \dots, x_r}\right) = \mathfrak{I}(x) = \mathfrak{I}\frac{\mathfrak{I}(t_0), \dots, \mathfrak{I}(t_r)}{x_0, \dots, x_r}(x) = \mathfrak{I}\frac{\mathfrak{I}(t_0), \dots, \mathfrak{I}(t_r)}{x_0, \dots, x_r}(t).$$

Otherwise, $x = x_i$ for some $0 \leqslant i \leqslant r$. Then $t\frac{t_0,\dots,t_r}{x_0,\dots,x_r} = t_i$. It follows that

$$\mathfrak{I}\left(t\frac{t_0,\dots,t_r}{x_0,\dots,x_r}\right) = \mathfrak{I}(t_i) = \mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r}(x_i) = \mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r}(t).$$

The other cases of t can be shown similarly.

(b) Assume that $\varphi = Rt'_1 \dots t'_n$. Then

$$\begin{aligned}
\mathfrak{I} \models \varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r} &\iff \left(\mathfrak{I}\Big(t'_1\frac{t_0,\dots,t_r}{x_0,\dots,x_r}\Big),\dots,\mathfrak{I}\Big(t'_n\frac{t_0,\dots,t_r}{x_0,\dots,x_r}\Big)\right) \in R^{\mathfrak{A}} \\
&\iff \left(\mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r}(t'_1),\dots,\mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r}(t'_n)\right) \in R^{\mathfrak{A}} \qquad \text{(by (a))} \\
&\iff \mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r} \models Rt'_1 \dots t'_n \\
&\qquad \text{i.e., } \mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r} \models \varphi.
\end{aligned}$$

For another case, let $\varphi = \exists x\psi$. Again, let $x_{i_1},\dots,x_{i_s}$ be the variables x_i with $x_i \in \text{free}(\exists x\psi)$ and $x_i \neq t_i$. Choose u according to Definition 1.6 (e). In particular, u does not occur in $t_{i_1},\dots,t_{i_s}$. Then

$$\begin{aligned}
\mathfrak{I} \models \varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r} &\iff \mathfrak{I} \models \exists u\left[\psi\frac{t_{i_1},\dots,t_{i_s},u}{x_{i_1},\dots,x_{i_s},x}\right] \\
&\iff \text{there exists an } a \in A \text{ such that } \mathfrak{I}\frac{a}{u} \models \psi\frac{t_{i_1},\dots,t_{i_s},u}{x_{i_1},\dots,x_{i_s},x} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathfrak{I}\frac{a}{u}\right]\frac{\mathfrak{I}\frac{a}{u}(t_{i_1}),\dots,\mathfrak{I}\frac{a}{u}(t_{i_s}),\mathfrak{I}\frac{a}{u}(u)}{x_{i_1},\dots,x_{i_s},x} \models \psi \\
&\qquad \text{(by induction hypothesis)} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathfrak{I}\frac{a}{u}\right]\frac{\mathfrak{I}(t_{i_1}),\dots,\mathfrak{I}(t_{i_s}),a}{x_{i_1},\dots,x_{i_s},x} \models \psi \\
&\qquad \text{(by the coincidence lemma and that } u \text{ does not occur in } t_{i_1},\dots t_{i_s}) \\
&\iff \text{there exists an } a \in A \text{ such that } \mathfrak{I}\frac{\mathfrak{I}(t_{i_1}),\dots,\mathfrak{I}(t_{i_s}),a}{x_{i_1},\dots,x_{i_s},x} \models \psi \\
&\qquad \text{(by (either } u = x \text{ or } u \text{ does not occur in } \psi) \text{ and the coincidence lemma)} \\
&\iff \text{there exists an } a \in A \text{ such that } \left[\mathfrak{I}\frac{\mathfrak{I}(t_{i_1}),\dots,\mathfrak{I}(t_{i_s})}{x_{i_1},\dots,x_{i_s}}\right]\frac{a}{x} \models \psi \\
&\qquad \text{(since } x \neq x_{i_1},\dots,x \neq x_{i_s}) \\
&\iff \mathfrak{I}\frac{\mathfrak{I}(t_{i_1}),\dots,\mathfrak{I}(t_{i_s})}{x_{i_1},\dots,x_{i_s}} \models \exists x\psi \\
&\iff \mathfrak{I}\frac{\mathfrak{I}(t_0),\dots,\mathfrak{I}(t_r)}{x_0,\dots,x_r} \models \exists x\psi \\
&\qquad \text{(by } x_i \notin \text{free}(\exists x\psi) \text{ or } x_i = t_i \text{ for } i \neq i_1,\dots,i \neq i_s).
\end{aligned}$$

□