Mathematical Logic (VII)

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1 Completeness

The goal of this section is to show:

Theorem 1.1 (Completeness). *If* $\Phi \models \varphi$*, then* $\Phi \vdash \varphi$. $\dashv$

We observe that the contrapositive of Theorem 1.1 is:

$$\Phi \nvdash \varphi \text{ implies } \Phi \not\models \varphi$$

$$\iff \text{if } \Phi \cup \{\neg\varphi\} \text{ is consistent, then } \Phi \cup \{\neg\varphi\} \text{ is satisfiable.}$$

As a matter of fact, we actually will prove the following general statement.

Theorem 1.2. cons(Φ) *implies that* Φ *is satisfiable.* $\dashv$

1.1 Henkin's Theorem

Recall that we fix a set Φ of S-formulas.

Definition 1.3. Let $t_1, t_2 \in T^S$. Then $t_1 \sim t_2$ if $\Phi \vdash t_1 \equiv t_2$. $\dashv$

Lemma 1.4. *(i)* $\sim$ *is an **equivalence** relation.*

(ii) $\sim$ *is a **congruence** relation. That is:*

- *For every* n*-ary function symbol* $R \in S$ *and* $2 \cdot n$ *S-terms* $t_1 \sim t_1', \ldots, t_n \sim t_n'$*, we have*

$$ft_1 \cdots t_n \sim ft_1' \cdots t_n'.$$

- *For every* n*-ary relation symbol* $R \in S$ *and* $2 \cdot n$ *S-terms* $t_1 \sim t_1', \ldots, t_n \sim t_n'$*, we have*

$$\Phi \vdash Rt_1 \cdots t_n \iff \Phi \vdash Rt_1' \cdots t_n'.$$ $\dashv$

Proof: By the equality rule and the substitution rule. □

Now for every $t \in T^S$ we define

$$\bar{t} := \{t' \in T^S \mid t' \sim t\},$$

i.e., the equivalence class of t.

Definition 1.5. The **term structure for** Φ, denoted by $\mathfrak{T}^\Phi$, is defined as below.

(i) The universe is $T^\Phi := \{\bar{t} \mid t \in T^S\}$.

(ii) For every n-ary relation symbol $R \in S$, and $\bar{t}_1, \ldots, \bar{t}_n \in T^\Phi$

$$(\bar{t}_1, \ldots, \bar{t}_n) \in R^{\mathfrak{T}^\Phi} \quad \text{if} \quad \Phi \vdash Rt_1 \ldots t_n.$$

(iii) For every n-ary function symbol $f \in S$, and $\bar{t}_1, \dots, \bar{t}_n \in T^\Phi$

$$f^{\mathfrak{T}^\Phi}(\bar{t}_1, \dots, \bar{t}_n) := \overline{ft_1 \cdots t_n}.$$

(iv) For every constant $c \in S$

$$c^{\mathfrak{T}^\Phi} := \bar{c}.$$

This finishes the construction of $\mathfrak{T}^\Phi$. ⊣

Using Lemma 1.4, in particular (ii), it is easy to verify that:

Lemma 1.6. $\mathfrak{T}^\Phi$ *is well-defined.* ⊣

To complete the definition of an S-interpretation, we still need to provide an assignment of the variables $v_0, v_1, \dots$ in $\mathfrak{T}^\Phi$.

Definition 1.7. For every variable v_i we let

$$\beta^\Phi(v_i) := \bar{v}_i.$$ ⊣

Finally we let

$$\mathfrak{I}^\Phi := \left(\mathfrak{T}^\Phi, \beta^\Phi\right).$$

Lemma 1.8. *(i) For any* $t \in T^S$

$$\mathfrak{I}^\Phi(t) = \bar{t}.$$

(ii) For every ***atomic*** φ

$$\mathfrak{I}^\Phi \models \varphi \quad \Longleftrightarrow \quad \Phi \vdash \varphi.$$ ⊣

Proof: (i) We proceed by induction on t.

- $t = v_i$ is a variable. Then
$$\mathfrak{I}^\Phi(v_i) = \beta^\Phi(v_i) = \bar{v}_i.$$
- $t = c$ is a constant. Then
$$\mathfrak{I}^\Phi(c) = c^{\mathfrak{T}^\Phi} = \bar{c}$$
- $t = ft_1 \cdots t_n$. Then
$$\begin{aligned} \mathfrak{I}^\Phi(ft_1 \cdots t_n) &= f^{\mathfrak{T}^\Phi}(\mathfrak{I}^\Phi(t_1), \dots, \mathfrak{I}^\Phi(t_n)) \\ &= f^{\mathfrak{T}^\Phi}(\bar{t}_1, \dots, \bar{t}_n) && \text{(by induction hypothesis)} \\ &= \overline{ft_1 \cdots t_n}. \end{aligned}$$

(ii) Recall that there are two types of atomic formulas. For the first type, let $\varphi = t_1 \equiv t_2$. Then

$$\begin{aligned} \mathfrak{I}^\Phi \models t_1 \equiv t_2 &\iff \mathfrak{I}^\Phi(t_1) = \mathfrak{I}^\Phi(t_2) \\ &\iff \bar{t}_1 = \bar{t}_2 && \text{(by (i))} \\ &\iff t_1 \sim t_2 \\ &\iff \Phi \vdash t_1 \equiv t_2. \end{aligned}$$

Second, let $\varphi = Rt_1 \cdots t_n$. We deduce

$$\begin{aligned} \mathfrak{I}^\Phi \models Rt_1 \cdots t_n &\iff \left(\mathfrak{I}^\Phi(t_1), \dots, \mathfrak{I}^\Phi(t_n)\right) \in R^{\mathfrak{T}^\Phi} \\ &\iff \left(\bar{t}_1, \dots, \bar{t}_n\right) \in R^{\mathfrak{T}^\Phi} && \text{(by (i))} \\ &\iff \Phi \vdash Rt_1 \cdots t_n. \end{aligned}$$

□

Lemma 1.9. *Let φ be an S-formula and $x_1, \ldots, x_n$ pairwise distinct variables. Then*

(i) $\mathfrak{I}^\Phi \models \exists x_1 \ldots \exists x_n \varphi$ *if and only if there are S-terms* $t_1, \ldots, t_n$ *such that*

$$\mathfrak{I}^\Phi \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n}.$$

(ii) $\mathfrak{I}^\Phi \models \forall x_1 \ldots \forall x_n \varphi$ *if and only if for all S-terms* $t_1, \ldots, t_n$ *we have*

$$\mathfrak{I}^\Phi \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n}.$$

Proof: We prove (i), then (ii) follows immediately.

$$\begin{aligned} \mathfrak{I}^\Phi \models & \exists x_1 \ldots \exists x_n \varphi \\ & \iff \mathfrak{I}^\Phi \frac{a_1 \ldots a_n}{x_1 \ldots x_n} \models \varphi \text{ for some } a_1, \ldots, a_n \in T^\Phi, \\ & \quad \text{i.e., } \mathfrak{I}^\Phi \frac{\bar{t}_1 \ldots \bar{t}_n}{x_1 \ldots x_n} \models \varphi \text{ for some } t_1, \ldots, t_n \in T^S, \\ & \iff \mathfrak{I}^\Phi \frac{\mathfrak{I}^\Phi(t_1) \ldots \mathfrak{I}^\Phi(t_n)}{x_1 \ldots x_n} \models \varphi \text{ for some } t_1, \ldots, t_n \in T^S, \qquad \text{(by Lemma 1.8 (i))} \\ & \iff \mathfrak{I}^\Phi \models \varphi \frac{t_1 \ldots t_n}{x_1 \ldots x_n} \text{ for some } t_1, \ldots, t_n \in T^S, \qquad \text{(by the Substitution Lemma).} \end{aligned}$$

□

Definition 1.10. (i) Φ is **negation complete** if for every S-formula φ

$$\Phi \vdash \varphi \quad \text{or} \quad \Phi \vdash \neg\varphi.$$

(ii) Φ contains witnesses if for every S-formula φ and every variable x there is a term $t \in T^S$ with

$$\Phi \vdash \left(\exists x \varphi \to \varphi \frac{t}{x} \right).$$

⊣

Lemma 1.11. *Assume that Φ is consistent, negation complete, and contains witnesses. Then for all S-formulas φ and ψ:*

(i) $\Phi \vdash \varphi$ *if and only if* $\Phi \nvdash \neg\varphi$.

(ii) $\Phi \vdash (\varphi \vee \psi)$ *if and only if* $\Phi \vdash \varphi$ *or* $\Phi \vdash \psi$.

(iii) $\Phi \vdash \exists x \varphi$ *if and only if there is a term* $t \in T^s$ *such that* $\Phi \vdash \varphi \frac{t}{x}$.

Proof: (i) Assume that $\Phi \vdash \varphi$. Since Φ is consistent, we conclude that $\Phi \nvdash \neg\varphi$. Conversely, if $\Phi \nvdash \neg\varphi$, then $\Phi \vdash \varphi$ by the negation completeness.

(ii) The direction from right to left is trivial by $\vee$-introduction in succedent. For the other direction, assume that $\Phi \vdash (\varphi \vee \psi)$ and $\Phi \nvdash \varphi$. By the negation completeness, $\Phi \vdash \neg\varphi$. Then for some finite $\Gamma \subseteq \Phi$ we have the following sequent proof.

	$\vdots$	
m.	$\Gamma_1 \quad (\varphi \vee \psi)$	
	$\vdots$	
n.	$\Gamma_2 \quad \neg\varphi$	
$(n+1)$.	$\Gamma_1 \quad \Gamma_2 \quad \varphi \quad \neg\varphi$	(antecedent by n)
$(n+2)$.	$\Gamma_1 \quad \Gamma_2 \quad \varphi \quad \varphi$	(assumption)
$(n+3)$.	$\Gamma_1 \quad \Gamma_2 \quad \varphi \quad \psi$	(modified contradiction by $n+1$ and $n+2$)
$(n+4)$.	$\Gamma_1 \quad \Gamma_2 \quad \psi \quad \psi$	(assumption)
$(n+5)$.	$\Gamma_1 \quad \Gamma_2 \quad (\varphi \vee \psi) \quad \psi$	($\vee$-introduction in antecedent)
$(n+6)$.	$\Gamma_1 \quad \Gamma_2 \quad \psi$	(chain rule by m and $n+5$)

(iii) Let $\Phi \vdash \exists x\varphi$ and Φ contain witnesses. Thus there is a term $t \in T^S$ such that

$$\Phi \vdash \left(\exists x\varphi \to \varphi\frac{t}{x}\right).$$

By Modus ponens[1], we conclude $\Phi \vdash \varphi\frac{t}{x}$. The converse is by the rule of the $\exists$-introduction in succedent. □

Theorem 1.12 (Henkin's Theorem). *Let $\Phi \subseteq L^S$ be consistent, negation complete, and contain witnesses. Then for every S-formula φ*

$$\mathfrak{I}^\Phi \models \varphi \iff \Phi \vdash \varphi.$$

Proof: We proceed by induction on φ.

- φ is atomic. This is Lemma 1.8 (ii).
- $\varphi = \neg\psi$. Then

$$\begin{aligned} \mathfrak{I}^\Phi \models \neg\psi &\iff \mathfrak{I}^\Phi \not\models \psi \\ &\iff \Phi \nvdash \psi && \text{(by induction hypothesis)} \\ &\iff \Phi \vdash \neg\psi && \text{(by Lemma 1.11 (i)).} \end{aligned}$$

- $\varphi = (\psi_1 \vee \psi_2)$. We deduce

$$\begin{aligned} \mathfrak{I}^\Phi \models (\psi_1 \vee \psi_2) &\iff \mathfrak{I}^\Phi \models \psi_1 \text{ or } \mathfrak{I}^\Phi \models \psi_2 \\ &\iff \Phi \vdash \psi_1 \text{ or } \Phi \vdash \psi_2 && \text{(by induction hypothesis)} \\ &\iff \Phi \vdash (\psi_1 \vee \psi_2) && \text{(by Lemma 1.11 (ii)).} \end{aligned}$$

- $\varphi = \exists x\psi$.

$$\begin{aligned} \mathfrak{I}^\Phi \models \exists x\psi &\iff \mathfrak{I}^\Phi \models \psi\frac{t}{x} \text{ for some } t \in T^S && \text{(by Lemma 1.9)} \\ &\iff \Phi \vdash \psi\frac{t}{x} \text{ for some } t \in T^S && \text{(by induction hypothesis)} \\ &\iff \Phi \vdash \exists x\psi && \text{(by Lemma 1.11 (iii)).} \end{aligned}$$

  Here, note that the length of $\psi\frac{t}{x}$ could be well larger than that $\exists x\psi$. Thus, our induction is on the so-called **connective rank** of ψ, denoted by $\mathrm{rk}(\varphi)$, which is defined as follows:

$$\mathrm{rk}(\varphi) := \begin{cases} 0 & \text{if } \varphi \text{ is atomic,} \\ 1 + \mathrm{rk}(\psi) & \text{if } \varphi = \neg\psi, \\ 1 + \mathrm{rk}(\psi_1) + \mathrm{rk}(\psi_2) & \text{if } \varphi = (\psi_1 \vee \psi_2), \\ 1 + \mathrm{rk}(\psi) & \text{if } \varphi = \exists x\psi. \end{cases}$$

□

Corollary 1.13. *Let $\Phi \subseteq L^S$ be consistent, negation complete, and contain witnesses. Then*

$$\mathfrak{I}^\Phi \models \Phi.$$

In particular, Φ is satisfiable.

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[1]That is, if $\Phi \vdash \varphi$ and $\Phi \vdash \varphi \to \psi$, then $\Phi \vdash \psi$.