Mathematical Logic (VIII)

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1 Completeness

1.1 Henkin's Theorem

Recall that we fix a set Φ of S-formulas.

Definition 1.1. (i) Φ is **negation complete** if for every S-formula φ

$$\Phi \vdash \varphi \quad \text{or} \quad \Phi \vdash \neg\varphi.$$

(ii) Φ **contains witnesses** if for every S-formula φ and every variable x there is a term $t \in T^S$ with

$$\Phi \vdash \left(\exists x\varphi \to \varphi\frac{t}{x}\right). \qquad \dashv$$

Theorem 1.2 (Henkin's Theorem)**.** *Let $\Phi \subseteq L^S$ be consistent, negation complete, and contain witnesses. Then for every S-formula φ*

$$\mathfrak{I}^\Phi \models \varphi \quad \iff \quad \Phi \vdash \varphi.$$

Corollary 1.3. *Let $\Phi \subseteq L^S$ be consistent, negation complete, and contain witnesses. Then*

$$\mathfrak{I}^\Phi \models \Phi.$$

In particular, Φ is satisfiable.

1.2 The countable case

We fix a symbol set S which is at most countable. As a consequence, both T^S and L^S are countable. Let $\Phi \subseteq L^S$ we define

$$\text{free}(\Phi) := \bigcup_{\varphi\in\Phi} \text{free}(\varphi).$$

We will prove the following two lemmas.

Lemma 1.4. *Let $\Phi \subseteq L^S$ be consistent with* ***finite*** free(Φ)*. Then there is a consistent Ψ with $\Phi \subseteq \Psi \subseteq L^S$ such that Ψ contains witnesses.*

Lemma 1.5. *Let $\Psi \subseteq L^S$ be consistent. Then there is a consistent Θ with $\Psi \subseteq \Theta \subseteq L^S$ such that Θ is negation complete.*

Corollary 1.6. *Let $\Phi \subseteq L^S$ be consistent with finite* free(Φ)*. Then there is a Θ such that*

- *$\Phi \subseteq \Theta \subseteq L^S$;*
- *Θ is consistent, negation complete, and contains witnesses.*

Corollary 1.7. *Let $\Phi \subseteq L^S$ be consistent with finite* $\mathrm{free}(\Phi)$*. Then Φ is satisfiable.*

Proof: By Corollary 1.6 and Corollary 1.3. $\square$

Proof of Lemma 1.4: Recall L^S is countable, thus we can enumerate all S-formulas

$$\exists x_0 \varphi_0, \exists x_1 \varphi_1, \dots,$$

which start with an existential quantifier. Then we define inductively for every $n \in \mathbb{N}$ an S-formula ψ_n as follows. Assume that ψ_m has been defined for all $m < n$. Let

$$i_n := \min\bigl\{i \in \mathbb{N} \bigm| v_i \notin \mathrm{free}\bigl(\Phi \cup \{\psi_m \mid m < n\} \cup \{\exists x_n \varphi_n\}\bigr)\bigr\}.$$

That is, i_n is the smallest index i such that v_i is not free in $\Phi \cup \{\psi_m \mid m < n\} \cup \{\exists x_n \varphi_n\}$. Then we set

$$\psi_n := \left(\exists x_n \varphi_n \to \varphi_n \frac{v_{i_n}}{x_n}\right).$$

Next, let

$$\Phi_n := \Phi \cup \bigl\{\psi_m \bigm| m < n\bigr\},$$

and $\Psi := \bigcup_{n \in \mathbb{N}} \Phi_n$. It should be clear that Φ contains witness. So what remains is to show that Ψ is consistent, or equivalently every Φ_n is consistent.

Recall that $\Phi_0 = \Phi$ is consistent by our assumption. Towards a contradiction, assume that Φ_n is consistent, but Φ_{n+1} is not. Therefore, for every χ with $v_{i_n} \notin \mathrm{free}(\chi)$ there is a finite $\Gamma \subseteq \Phi_n$ with the following deduction.

	$\vdots$			
$m.$	Γ	$\left(\neg\exists x_n \varphi_n \vee \varphi_n \frac{v_{i_n}}{x_n}\right)$	χ	
$(m+1).$	Γ	$\neg\exists x_n \varphi_n$	$\neg\exists x_n \varphi_n$	(assumption)
$(m+2).$	Γ	$\neg\exists x_n \varphi_n$	$\left(\neg\exists x_n \varphi_n \vee \varphi_n \frac{v_{i_n}}{x_n}\right)$	($\vee$-introduction in the succedent)
$(m+3).$	Γ	$\neg\exists x_n \varphi_n$	χ	(chain rule)
	$\vdots$			
$(\ell).$	Γ	$\varphi_n \frac{v_{i_n}}{x_n}$	χ	(similarly)
$(\ell+1).$	Γ	$\exists x_n \varphi_n$	χ	($\exists$-introduction in the antecedent)
$(\ell+2).$	Γ		χ	(case analysis).

Now by taking $\chi := \exists v_0 v_0 \equiv v_0$ and $\chi := \neg\exists v_0 v_0 \equiv v_0$ we conclude that Φ_n is inconsistent, which contradicts our assumption. $\square$

Proof of Lemma 1.5: Let $\varphi_0, \varphi_1, \dots$ be an enumeration of L^S. For every $n \in \mathbb{N}$ we define Θ_n by induction. First $\Theta_0 := \Psi$. Then,

$$\Theta_{n+1} := \begin{cases} \Theta_n \cup \{\varphi_n\} & \text{if } \Theta_n \cup \{\varphi_n\} \text{ is consistent,} \\ \Theta_n & \text{otherwise.} \end{cases}$$

It is immediate that every Θ_n is consistent, and the consistency of

$$\Theta := \bigcup_{n \in \mathbb{N}} \Theta_n$$

follows. To see that Θ is negation complete, let $\varphi \in L^S$, in particular $\varphi = \varphi_n$ for some $n \in \mathbb{N}$. Assuming $\Theta \nvdash \neg\varphi_n$, we conclude $\Theta_n \nvdash \neg\varphi_n$ by $\Theta_n \subseteq \Theta$. Therefore, $\Theta_n \cup \{\varphi\}$ is consistent. It follows that $\varphi \in \Theta_{n+1} \subseteq \Theta$, and thus $\Theta \vdash \varphi$. $\square$

In the next step we eliminate the condition free(Φ) being finite.

Corollary 1.8. *Let S be countable and $\Phi \subseteq L^S$ consistent. Then Φ is satisfiable.*

Proof: First, we let

$$S' := S \cup \{c_0, c_1, \dots\}.$$

For every $\varphi \in L^S$ we define

$$n(\varphi) := \min\big\{n \;\big|\; \text{free}(\varphi) \subseteq \{v_0, \dots, v_{n-1}\}, \text{ i.e., } \varphi \in L^S_n\big\},$$

and let

$$\varphi' := \varphi \frac{c_0 \dots c_{n(\varphi)-1}}{v_0 \dots v_{n(\varphi)-1}}.$$

Then we set

$$\Phi' := \big\{\varphi' \;\big|\; \varphi \in \Phi\big\} \subseteq L^{S'}$$

Note free$(\Phi') = \emptyset$.

Claim. Φ' is consistent.

Once we establish the claim, together with free$(\Phi') = \emptyset$, Corollary 1.6 implies that there is an S'-interpretation $\mathfrak{I}' = (\mathfrak{A}', \beta')$ such that $\mathfrak{I}' \models \Phi'$. Applying the Coincidence Lemma with free$(\Phi') = \emptyset$, we can assume without loss of generality that

$$\beta'(v_i) = c_i^{\mathfrak{A}'} = \mathfrak{I}'(c_i). \tag{1}$$

It follows that for every $\varphi \in \Phi$

$$\begin{aligned}
\mathfrak{I}' \models \varphi' &\iff \mathfrak{I}' \models \varphi \frac{c_0 \dots c_{n(\varphi)-1}}{v_0 \dots v_{n(\varphi)-1}} \\
&\iff \mathfrak{I}' \frac{\mathfrak{I}'(c_0) \dots \mathfrak{I}'(c_{n(\varphi)-1})}{v_0 \dots v_{n(\varphi)-1}} \models \varphi && \text{(by the Substitution Lemma)} \\
&\iff \mathfrak{I}' \frac{\beta'(v_0) \dots \beta'(v_{n(\varphi)-1})}{v_0 \dots v_{n(\varphi)-1}} \models \varphi && \text{(by (1))} \\
&\quad\;\text{i.e., } \mathfrak{I}' \models \varphi.
\end{aligned}$$

That is, $\mathfrak{I}'$ is a model for every $\varphi \in \Phi$. We conclude that Φ is satisfiable.

Now we prove the claim. It suffices to show that every finite subset of Φ' is satisfiable. To that end, let

$$\Phi'_0 := \big\{\varphi'_1, \dots, \varphi'_n\big\},$$

where $\varphi_1, \dots, \varphi_n \in \Phi$. Clearly free$\big(\{\varphi_1, \dots, \varphi_n\}\big)$ is finite, and $\{\varphi_1, \dots, \varphi_n\}$ is consistent by the consistency of Φ. By Corollary 1.6 there is an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ such that for every $i \in [n]$

$$\mathfrak{I} \models \varphi_i. \tag{2}$$

We expand the S-structure $\mathfrak{A}$ to an S'-structure $\mathfrak{A}'$ by setting for every $i \in \mathbb{N}$

$$c_i^{\mathfrak{A}'} := \beta(v_i). \tag{3}$$

Then for the S'-interpretation $\mathfrak{I}' := (\mathfrak{A}', \beta)$ and any $\varphi \in L^S$

$$\begin{aligned}
\mathfrak{I}' \models \varphi' &\iff \mathfrak{I}' \models \varphi\frac{c_0 \ldots c_{n(\varphi)-1}}{v_0 \ldots v_{n(\varphi)-1}} \\
&\iff \mathfrak{I}'\frac{\mathfrak{I}'(c_0) \ldots \mathfrak{I}'(c_{n(\varphi)-1})}{v_0 \ldots v_{n(\varphi)-1}} \models \varphi && \text{(by the Substitution Lemma)} \\
&\iff \mathfrak{I}'\frac{c_0^{\mathfrak{A}'} \ldots c_{n(\varphi)-1}^{\mathfrak{A}'}}{v_0 \ldots v_{n(\varphi)-1}} \models \varphi \\
&\iff \mathfrak{I}'\frac{\beta(v_0) \ldots \beta(v_{n(\varphi)-1})}{v_0 \ldots v_{n(\varphi)-1}} \models \varphi && \text{(by (3))} \\
&\iff \mathfrak{I}' \models \varphi \\
&\iff \mathfrak{I} \models \varphi && \text{(by the Coincidence Lemma).}
\end{aligned}$$

It follows that $\mathfrak{I}' \models \Phi_0'$ by (2). Thus Φ_0' is satisfiable. $\square$