Mathematical Logic (IX)

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1. Completeness

Recall that we have shown:

Lemma 1.1. *Let* $\Phi \subseteq L^S$ *and* $\mathfrak{I}^\Phi$ *be the term interpretation of* Φ*. Then for every* atomic φ

$$\mathfrak{I}^\Phi \models \varphi \iff \Phi \vdash \varphi. \qquad \dashv$$

Theorem 1.2 (Henkin's Theorem)**.** *Let* $\Phi \subseteq L^S$ *be consistent, negation complete, and contain witnesses. Then for every S-formula* φ

$$\mathfrak{I}^\Phi \models \varphi \iff \Phi \vdash \varphi. \qquad \dashv$$

Corollary 1.3. *Let S be countable and* $\Phi \subseteq L^S$ *consistent with finite* free(Φ)*. Then there is a* Θ *such that*

- $\Phi \subseteq \Theta \subseteq L^S$;
- Θ *is consistent, negation complete, and contains witnesses.*

Therefore by Theorem 1.2 for every $\varphi \in L^S$

$$\mathfrak{I}^\Theta \models \varphi \iff \Theta \vdash \varphi.$$

In particular

$$\mathfrak{I}^\Theta \models \Phi,$$

thus Φ *is satisfiable.* $\dashv$

In the next step we eliminate the condition free(Φ) being finite.

Corollary 1.4. *Let S be countable and* $\Phi \subseteq L^S$ *consistent. Then* Φ *is satisfiable.*

1.1. The general case.

Lemma 1.5. *Let* $\Phi \subseteq L^S$ *be consistent. Then there is a symbol set* S' *with* $S \subseteq S'$ *and a consistent* Ψ *with* $\Phi \subseteq \Psi \subseteq L^{S'}$ *such that* Ψ *contains witnesses.* $\dashv$

Lemma 1.6. *Let* $\Psi \subseteq L^S$ *be consistent. Then there is a consistent* Θ *with* $\Psi \subseteq \Theta \subseteq L^S$ *such that* Θ *is negation complete.* $\dashv$

Then the next corollary follows from Lemmas 1.5 and 1.6 in the same fashion as that of Corollary 1.3.

Corollary 1.7. *Let* $\Phi \subseteq L^S$ *be consistent. Then* Φ *is satisfiable.* $\dashv$

We need some technical tools for proving Lemma 1.5. Let S be an arbitrary symbol set. For every $\varphi \in L^S$ we introduce a constant $c_\varphi \notin S$. In particular, $c_\varphi \neq c_\psi$ for any $\varphi \neq \psi$. Then we set

$$S^* := S \cup \{c_{\exists x\varphi} \mid \exists x\varphi \in L^S\},$$

$$W(S) := \left\{\exists x\varphi \to \varphi\frac{c_{\exists x\varphi}}{x} \;\middle|\; \exists x\varphi \in L^S\right\}.$$

It is obvious that $c_{\exists x\varphi}$ is introduced as a witness for $\exists x\varphi$ as required by $W(S)$. Nevertheless, we pay a price for expanding the symbol set S to S^*, i.e., there are formulas of the form $\exists x\varphi$ in $L^{S^*} \setminus L^S$, e.g.,

$$\exists v_7 c_{\exists x Rx} \equiv v_7.$$

Lemma 1.8. *Assume that $\Phi \subseteq L^S$ is consistent. Then*

$$\Phi \cup W(S) \subseteq L^{S^*}$$

is consistent as well.

Proof: It suffices to show that every finite subset Φ_0^* of $\Phi \cup W(S) \subseteq L^{S^*}$ is satisfiable. Let

$$\Phi_0^* = \Phi_0 \cup \left\{\exists x_1\varphi_1 \to \varphi_1\frac{c_1}{x_1}, \ldots, \exists x_n\varphi_n \to \varphi_n\frac{c_n}{x_n}\right\},$$

where $\Phi_0 \subseteq \Phi$ is finite, every $\exists x_i\varphi_i \in L^S$, and $c_i = c_{\exists x_i\varphi_i}$ for $i \in [n]$.

Choose a finite $S_0 \subseteq S$ such that $\Phi_0 \subseteq L^{S_0}$. Note that Φ_0 is consistent due to the consistency of Φ. Furthermore free(Φ_0) is finite[1]. Therefore Φ_0 is satisfiable by Corollary 1.3, i.e., there is an S_0-interpretation $\mathfrak{I}_0 = (\mathfrak{A}_0, \beta)$ such that

$$\mathfrak{I}_0 \models \Phi_0$$

Note that $\mathfrak{A}_0$ is an S_0-structure. By choosing some arbitrary interpretation of the symbols in $S \setminus S_0$ we obtain an S-structure $\mathfrak{A}$. Then the Coincidence Lemma guarantees that for the S-interpretation $\mathfrak{I} := (\mathfrak{A}, \beta)$

$$\mathfrak{I} \models \Phi_0.$$

Next, we need to further expand $\mathfrak{A}$ to an S^*-structure $\mathfrak{A}^*$ by giving interpretation of all new constants $c_{\exists x\varphi}$. Let $a \in A$ be an arbitrary but fixed element. Then for every $i \in [n]$ we set

$$c_i^{\mathfrak{A}^*} := \begin{cases} a_i & \text{if there is an } a_i \in A \text{ with } \mathfrak{I} \models \varphi_i\frac{a_i}{x_i}, \\ & \quad \text{(choose an arbitrary one, if there are more than one such } a_i), \\ a & \text{otherwise.} \end{cases}$$

For all the other new constants $c_{\exists x\varphi}$ we simply let $c_{\exists x\varphi}^{\mathfrak{A}^*} := a$. Then for the S^*-interpretation $\mathfrak{I}^* := (\mathfrak{A}^*, \beta)$ we claim

$$\mathfrak{I}^* \models \Phi_0 \cup \left\{\exists x_1\varphi_1 \to \varphi_1\frac{c_1}{x_1}, \ldots, \exists x_n\varphi_n \to \varphi_n\frac{c_n}{x_n}\right\}.$$

$\mathfrak{I}^* \models \Phi_0$ is immediate by $\mathfrak{I} \models \Phi_0$ and the Coincidence Lemma. Let $i \in [n]$ and assume $\mathfrak{I}^* \models \exists x_i\varphi_i$, or equivalently $\mathfrak{I} \models \exists x_i\varphi_i$. Then by our choice of $a_i \in A$

$$\mathfrak{I} \models \varphi_i\frac{a_i}{x_i},$$

hence

$$\mathfrak{I}^* \models \exists x_i\varphi_i \to \varphi_i\frac{c_i}{x_i}, \tag{1}$$

by the Coincidence Lemma and by the Substitution Lemma. Note (1) trivially holds if $\mathfrak{I}^* \not\models \exists x_i\varphi_i$. This finishes the proof. □

[1] Here, we can also apply Corollary 1.4 without using the finiteness of free(Φ_0). But then this would introduce a further layer of construction as in the proof of Corollary 1.4.

Lemma 1.9. *Let*

$$S_0 \subseteq S_1 \subseteq \cdots \subseteq S_n \subseteq \cdots$$

be a sequence of symbol sets. Furthermore, for every $n \in \mathbb{N}$ *let* Φ_n *be a set of* S_n*-formulas such that*

$$\Phi_0 \subseteq \Phi_1 \subseteq \cdots \subseteq \Phi_n \subseteq \cdots$$

We set

$$S := \bigcup_{n \in \mathbb{N}} S_n \quad \textit{and} \quad \Phi := \bigcup_{n \in \mathbb{N}} \Phi_n.$$

Then Φ *is a consistent set of* S*-formulas if and only if every* Φ_n *is consistent.*

Proof: We prove that

$$\Phi \text{ is inconsistent} \quad \iff \quad \Phi_n \text{ is inconsistent for some } n \in \mathbb{N}.$$

The direction from right to left is trivial. So assume that Φ is inconsistent. In particular, for some $\varphi \in L^S$ there are proofs of φ and $\neg\varphi$ from Φ. Since proofs in sequent calculus are all finite, we can choose a finite $S' \subseteq S$ such that every formula used in the proofs of φ and $\neg\varphi$ is an S'-formula. For the same reason, for a sufficiently large $n \in \mathbb{N}$ we have

(i) $S' \subseteq S_n$,

(ii) $\Phi_n \vdash \varphi$ and $\Phi_n \vdash \neg\varphi$.

Thus Φ_n is inconsistent. □

Remark 1.10. Note at this point we have not shown the following seemingly trivial result. Let S be an (infinite) set of symbols, a finite $\Phi \subseteq L^S$, and $\varphi \in L^S$ such that $\Phi \vdash \varphi$. Furthermore, let $S_0 \subseteq S$ be the set of symbols that occur in Φ and φ. Then there is a proof of sequence calculus for $\Phi \vdash \varphi$ such that every formula occurs in the proof is an S_0-formula, i.e., only uses symbols in S_0.

This is the reason in the proof of Lemma 1.9 we need to emphasize (i). ⊣

Proof of Lemma 1.5: Let

$$S_0 := S \quad \text{and} \quad S_{n+1} := (S_n)^*,$$
$$\Psi_0 := \Phi \quad \text{and} \quad \Psi_{n+1} := \Psi_n \cup W(S_n).$$

Therefore

$$S = S_0 \subseteq \cdots \subseteq S_n \subseteq S_{n+1} \subseteq \cdots$$
$$\Phi = \Psi_0 \subseteq \cdots \subseteq \Psi_n \subseteq \Psi_{n+1} \subseteq \cdots$$

Then we set

$$S' := \bigcup_{n \in \mathbb{N}} S_n \quad \text{and} \quad \Psi := \bigcup_{n \in \mathbb{N}} \Psi_n.$$

By Lemma 1.8 and induction on n we conclude that every Ψ_n is consistent. Thus Lemma 1.9 implies that Ψ is a consistent set of S'-formulas.

By our construction of $W(S_n)$, the set Ψ trivially contains witnesses. □

The proof of Lemma 1.6 relies on well-known Zorn's Lemma. Let M be a set and $\mathcal{U} \subseteq \mathscr{P}ow(M) = \{T \mid T \subseteq M\}$. We say that a *nonempty* subset $C \subseteq \mathcal{U}$ is a *chain* in $\mathcal{U}$ if for every $T_1, T_2 \in C$ either $T_1 \subseteq T_2$ or $T_2 \subseteq T_1$.

Lemma 1.11 (Zorn's Lemma[2]). *Assume that for every chain* C *in* $\mathfrak{U}$ *we have*

$$\bigcup C := \{a \mid a \in T \text{ for some } T \in C\} \in \mathfrak{U}.$$

Then $\mathfrak{U}$ *has a* maximal element T*, i.e., there is no* $T' \in \mathfrak{U}$ *with* $T \subsetneq T'$. ⊣

Proof of Lemma 1.6 In order to apply Zorn's Lemma we let $M := L^S$ and

$$\mathfrak{U} := \{\Theta \mid \Psi \subseteq \Theta \subseteq L^S \text{ and } \Theta \text{ is consistent}\}.$$

Let C be a chain in $\mathfrak{U}$. We set

$$\Theta_C := \bigcup C = \{\varphi \mid \varphi \in \Theta \text{ for some } \Theta \in C\}.$$

$C \neq \emptyset$ implies $\Psi \subseteq \Theta_C$. To see that Θ_C is consistent, let $\{\varphi_1, \ldots, \varphi_n\}$ be a finite subset of Θ_C, in particular, there are $\Theta_i \in C$ such that $\varphi_i \in \Theta_i$. As C is a chain, without loss of generality, we can assume that every $\Theta_i \subseteq \Theta_n$. Since $\Theta_n \in C$ is consistent by the definition of $\mathfrak{U}$, we conclude $\{\varphi_1, \ldots, \varphi_n\}$ is consistent as well.

Thus the condition in Zorn's Lemma is satisfied. It follows that $\mathfrak{U}$ has a maximal element Θ. We claim that Θ is negation complete. Otherwise, for some $\varphi \in L^S$ we have $\Theta \nvdash \varphi$ and $\Theta \nvdash \neg\varphi$. Therefore $\varphi \notin \Theta$ and $\Theta \cup \{\varphi\}$ is consistent. As a consequence $\Theta \subsetneq \Theta \cup \{\varphi\} \in \mathfrak{U}$. This is a contradiction to the maximality of Θ. □

Now we are ready to prove the completeness theorem.

Theorem 1.12. *Let* $\Phi \subseteq L^S$ *and* $\varphi \in L^S$. *Then*

$$\Phi \vdash \varphi \quad \Longleftrightarrow \quad \Phi \models \varphi.$$

Proof: The direction from left to right is easy by the soundness of sequent calculus. Conversely, assume that $\Phi \nvdash \varphi$, then $\Phi \cup \{\neg\varphi\}$ is consistent. By Corollary 1.7, $\Phi \cup \{\neg\varphi\}$ is satisfiable. Then, there is an S-interpretation $\mathfrak{I}$ with $\mathfrak{I} \models \Phi$ and $\mathfrak{I} \models \neg\varphi$ (i.e., $\mathfrak{I} \not\models \varphi$). But this means that $\Phi \not\models \varphi$. □

2. The Löwenheim-Skolem Theorem and the Compactness Theorem

Using the term-interpretation, it is routine to verify:

Theorem 2.1 (Löwenheim-Skolem). *Let* $\Phi \subseteq L^S$ *be at most countable and satisfiable. Then there is an S-interpretation* $\mathfrak{I} = (\mathfrak{A}, \beta)$ *such that*

- *the universe* A *of* $\mathfrak{A}$ *is at most countable,*
- *and* $\mathfrak{I} \models \Phi$. ⊣

The following is a more general version.

Theorem 2.2 (Downward Löwenheim-Skolem). *Let* $\Phi \subseteq L^S$ *be satisfiable. Then there is an S-interpretation* $\mathfrak{I} = (\mathfrak{A}, \beta)$ *such that*

- $|A| \leqslant |T^S| = |L^S|$,
- *and* $\mathfrak{I} \models \Phi$. ⊣

Corollary 2.3. *Let* $S := \{+, \times, <, 0, 1\}$ *with the usual meaning and*

$$\Phi_{\mathbb{R}} := \{\varphi \in L_0^S \mid (\mathbb{R}, +, \cdot, <, 0, 1) \models \varphi\}.$$

Then there is a countable S-*structure* $\mathfrak{A}$ *with* $\mathfrak{A} \models \Phi_{\mathbb{R}}$. ⊣

[2] See Canvas for a proof of Zorn's Lemma.

Theorem 2.4 (Compactness). *(a)* $\Phi \models \varphi$ *if and only if there is a* finite $\Phi_0 \subseteq \Phi$ *with* $\Phi_0 \models \varphi$.

(b) Φ *is satisfiable if and only if every finite* $\Phi_0 \subseteq \Phi$ *is satisfiable.* $\dashv$

In fact, the "compactness" is a notion from topology. We can explain the topological perspective of Theorem 2.4 using *finite covers* from analysis. For every $\varphi \in L^S$ we define

$$\mathrm{Mod}(\varphi) := \{\mathfrak{I} \mid \mathfrak{I} \models \varphi\},$$

and

$$\mathrm{Mod}(\Phi) := \{\mathfrak{I} \mid \mathfrak{I} \models \Phi\} = \bigcap_{\psi \in \Phi} \mathrm{Mod}(\psi).$$

We show that Theorem 2.4 is equivalent to the following *finite cover property*.

Proposition 2.5. $\mathrm{Mod}(\varphi) \subseteq \bigcup_{\psi \in \Phi} \mathrm{Mod}(\psi)$ *if and only if for some finite* $\Phi_0 \subseteq \Phi$ *we have*

$$\mathrm{Mod}(\varphi) \subseteq \bigcup_{\psi \in \Phi_0} \mathrm{Mod}(\psi). \qquad \dashv$$

Proof of Theorem 2.4 using Proposition 2.5:

$$\begin{aligned}
\Phi \models \varphi &\iff \mathrm{Mod}(\Phi) \subseteq \mathrm{Mod}(\varphi) \\
&\iff \overline{\mathrm{Mod}(\varphi)} \subseteq \overline{\mathrm{Mod}(\Phi)} \\
&\iff \overline{\mathrm{Mod}(\varphi)} \subseteq \overline{\bigcap_{\psi \in \Phi} \mathrm{Mod}(\psi)} \\
&\iff \overline{\mathrm{Mod}(\varphi)} \subseteq \bigcup_{\psi \in \Phi} \overline{\mathrm{Mod}(\psi)} \\
&\iff \mathrm{Mod}(\neg\varphi) \subseteq \bigcup_{\psi \in \Phi} \mathrm{Mod}(\neg\psi) \\
&\iff \mathrm{Mod}(\neg\varphi) \subseteq \bigcup_{\psi \in \Phi_0} \mathrm{Mod}(\neg\psi) \text{ for some finite } \Phi_0 \subseteq \Phi \qquad \text{(by Proposition 2.5)} \\
&\iff \overline{\mathrm{Mod}(\varphi)} \subseteq \bigcup_{\psi \in \Phi_0} \overline{\mathrm{Mod}(\psi)} \text{ for some finite } \Phi_0 \subseteq \Phi \\
&\iff \overline{\mathrm{Mod}(\varphi)} \subseteq \overline{\bigcap_{\psi \in \Phi_0} \mathrm{Mod}(\psi)} \text{ for some finite } \Phi_0 \subseteq \Phi \\
&\iff \bigcap_{\psi \in \Phi_0} \mathrm{Mod}(\psi) \subseteq \mathrm{Mod}(\varphi) \text{ for some finite } \Phi_0 \subseteq \Phi \\
&\iff \mathrm{Mod}(\Phi_0) \subseteq \mathrm{Mod}(\varphi) \text{ for some finite } \Phi_0 \subseteq \Phi \\
&\iff \Phi_0 \models \varphi \text{ for some finite } \Phi_0 \subseteq \Phi.
\end{aligned}$$

□

Proof of Proposition 2.5 by Theorem 2.4: The direction from right to left is trivial. So we assume that

$$\mathrm{Mod}(\varphi) \subseteq \bigcup_{\psi \in \Phi} \mathrm{Mod}(\psi).$$

Claim. $\{\neg\psi \mid \psi \in \Phi\} \models \neg\varphi$.

Proof of the claim. Let $\mathfrak{I}$ be an interpretation with

$$\mathfrak{I} \models \{\neg\psi \mid \psi \in \Phi\}.$$

That is, $\mathfrak{I} \models \neg\psi$ for every $\psi \in \Phi$. We can deduce that

$$\begin{aligned}
\mathfrak{I} \in \bigcap_{\psi\in\Phi} \mathrm{Mod}(\neg\psi) &\iff \mathfrak{I} \in \bigcap_{\psi\in\Phi} \overline{\mathrm{Mod}(\psi)} \\
&\iff \mathfrak{I} \in \overline{\bigcup_{\psi\in\Phi} \mathrm{Mod}(\psi)} \\
&\iff \mathfrak{I} \notin \bigcup_{\psi\in\Phi} \mathrm{Mod}(\psi) \\
&\implies \mathfrak{I} \notin \mathrm{Mod}(\varphi) \qquad \left(\text{by } \mathrm{Mod}(\varphi) \subseteq \bigcup_{\psi\in\Phi} \mathrm{Mod}(\psi)\right) \\
&\iff \mathfrak{I} \models \neg\varphi.
\end{aligned}$$

This finishes the proof of the claim. ⊣

Now we apply Theorem 2.4 to the above claim. In particular, there is a finite $\Phi_0 \subseteq \Phi$ such that

$$\{\neg\psi \mid \psi \in \Phi_0\} \models \neg\varphi$$

Then arguing similarly as above, we obtain

$$\mathrm{Mod}(\varphi) \subseteq \bigcup_{\psi\in\Phi_0} \mathrm{Mod}(\psi).$$ □

Theorem 2.6. *Let $\Phi \subseteq L^S$ such that for every $n \in \mathbb{N}$ there exists an S-interpretation $\mathfrak{I}_n = (\mathfrak{A}_n, \beta_n)$ with $|A_n| \geqslant n$ and $\mathfrak{I}_n \models \Phi$. Then there is an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ with* infinite *A and $\mathfrak{I} \models \Phi$.*

Proof: For every $n \geqslant 2$ we define a sentence

$$\varphi_{\geqslant n} := \exists v_0 \cdots \exists v_{n-1} \bigwedge_{0\leqslant i<j\leqslant n} \neg v_i \equiv v_j.$$

Clearly for any structure $\mathfrak{A}$ (regardless of the symbol set S)

$$\mathfrak{A} \models \varphi_{\geqslant n} \quad\iff\quad |A| \geqslant n.$$

Now consider

$$\Psi := \Phi \cup \left\{\varphi_{\geqslant n} \;\middle|\; n \geqslant 2\right\}.$$

Of course every finite subset of Ψ is contained in

$$\Psi_{n_0} := \Phi \cup \left\{\varphi_{\geqslant n} \;\middle|\; 2 \leqslant n \leqslant n_0\right\}$$

for a *sufficiently large* $n_0 \in \mathbb{N}$. By assumption, $\mathfrak{I}_{n_0}$ witnesses that Ψ_{n_0} is satisfiable. Therefore, by the Compactness Theorem, Ψ itself is satisfiable. The result follows immediately. □

Theorem 2.7 (Upward Löwenheim-Skolem). *Let $\Phi \subseteq L^S$ and assume that there is an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ such that A is infinite and $\mathfrak{I} \models \Phi$. Then, for any set B there is an S-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$ with $|A| \geqslant |B|$ and $\mathfrak{I} \models \Phi$.*

Proof: For any $b \in B$ we introduce a new constant $c_b \notin S$. In particular, $c_b \neq c_{b'}$ for any $b, b' \in B$ with $b \neq b$. Then consider

$$\Psi := \Phi \cup \left\{\neg c_b \equiv c_{b'} \;\middle|\; b, b' \in B \text{ with } b \neq b'\right\}.$$

Since Φ has an infinite interpretation, every finite subset of Ψ is satisfiable. By the Compactness Theorem, we conclude that Φ is satisfiable. Clearly the structure in any interpretation which satisfies Ψ must have size as large as $|B|$. □

Corollary 2.8. *Let $S = \{+, \times, <, 0, 1\}$ and*

$$\Phi_{\mathbb{N}} := \left\{\varphi \in L_0^S \;\middle|\; (\mathbb{N}, +, \cdot, <, 0, 1) \models \varphi\right\}.$$

Then there is a uncountable *S-structure $\mathfrak{A}$ with $\mathfrak{A} \models \Phi_{\mathbb{N}}$.* ⊣