Capacity, Coverage and Connectivity of Wireless Networks

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***Abstract*—The throughput $\lambda(n)$ of an ad hoc system consisting of n static nodes randomly located in a disk of unit area is $\Theta\left(\frac{W}{\sqrt{n\log n}}\right)$, which converges to zero when number of nodes goes to infinity. Although mobility can increase the overall throughput to $\Theta(1)$, there still exists several cases that nodes might be impossible to be mobile.**

In order to compensate the decrease in throughput $\lambda(n)$, we add mobile relay nodes, which only transmit but never generate information, to an ad hoc system consisting of purely static nodes.

In this paper, we first study static network with infinite mobile relay nodes. Then we study static network with finite mobile relay nodes. It is shown that the throughput of a static network with infinite mobile relay nodes is $\Theta(1)$, and the throughput of a static network with finite mobile relay nodes varies with the number of mobile relay nodes.

***Index Terms*—Ad hoc networks, capacity, relay.**

I. Introduction

Wireless networks consist of a number of nodes which communicate with each other over a wireless channel. *Ad hoc* network is one type of wireless networks which includes no wired backbone or centralized controlling center. Each node transmits packets directly to its destination or through several relay nodes. One simple example of ad hoc network is collection of furnitures in buildings, including air conditioners, refridgerators, personal computers, microwave ovens, and possibly other "smart" furnitures.

It is proved that the throughput $\lambda(n)$ of an ad hoc network consisting of static nodes is $\Theta\left(\frac{W}{\sqrt{n\log n}}\right)$, see Gupta and Kumar [2]; and even under optimal circumstances, the throughput is only $\Theta\left(\frac{W}{\sqrt{n}}\right)$ for each node for a destination nonvarnishingly far away. Throughput per node of mobile ad hoc wireless network is proved to reach $\Theta(1)$, see Matthias Grossglauser and David N. C. Tse [3], which is a optimistic result. However, in some situations, like "smart home", where nodes are constituted of furnitures, it is impossible to force nodes to be mobile. Since the probability that throughput $\Theta\left(\frac{W}{\sqrt{n\log n}}\right)$ is feasible approaches 1 as $n\to\infty$, a constraint on number of nodes n is not acceptable.

One approach to compensate the decrease of throughput $\lambda(n)$ is to add some mobile relay nodes to a static ad hoc network. These relays nodes generate no information themselves, thus causing no transmission requirement for the overall network, and their only job is to transmit information between static source nodes and destination nodes. Thus a multi-hop protocol can be replicated by a two-hop protocol. Intuitively, this model would increase the throughput $\lambda(n)$ to a large extent. Yet a restriction on it is that the increase of number of nodes, including static nodes and mobile nodes, leads to increase in interference, which is the major restriction on the throughput of ad hoc networks.

II. Model

The ad hoc networks consists of n static nodes and m ($m\le n$) mobile relay nodes all lying in the disk of unit area (of radius $\frac{1}{\sqrt{\pi}}$). Static nodes are randomly located, i.e., independently and uniformly distributed. Each static node has a randomly chosen destination to which it wishes to send $\lambda(n)$ bits per second. Relay nodes generate no information for transmission. The location of the ith relay node at time t is given by $Y_i(t)$. Relay nodes are mobile and with infinite storage, and we assume that the process $\{Y_i(\cdot)\}$ is stationary and ergodic with stationary distribution uniform on the disk; moreover, the trajectories of different relays are independent and identically distributed (i.i.d.). The destination for each node is independently chosen as the static node nearest to a randomly located point, i.e., uniformly and independently distributed. (Thus destinations are on the order to 1 m away on average).

In a random setting, we will assume that the nodes are homogeneous, i.e., all transmissions employ the same nominal range or power. The *Protocol Model* and *Physical Model* are as follows.

1) *The Protocol Model:* All nodes employ a common *range* r for all their transmissions. When the node X_i transmits to a node X_j over the mth subchannel, this transmission is successfully received by X_j if
   a) The distance between X_i and X_j is no more than r, i.e.,
   $$|X_i - X_j| \le r. \tag{1}$$
   b) For every other node X_k simultaneously transmitting over the same subchannel
   $$|X_k - X_j| \ge (1+\Delta)\,r. \tag{2}$$
2) *The Physical Model:* All nodes choose a common power level P for all their transmissions. Let $\{X_k; k\in\mathcal{T}\}$ be the subset of nodes simultaneously transmitting at some instant over a certain subchannel. A transmission from a node $X_i,\ i\in\mathcal{T}$, is successfully received by a node

X_j if

$$\frac{\frac{P}{|X_i - X_j|^\kappa}}{N + \sum_{\substack{k \in \mathcal{T} \\ k \neq i}} \frac{P}{|X_k - X_j|^\kappa}} \geq \beta. \tag{3}$$

Denote by $\gamma_{i,j}$ the channel gain from node i to node j, Equation 3 can be rewritten as

$$\frac{P\gamma_{i,j}}{N + \sum_{\substack{k \in \mathcal{T} \\ k \neq i}} P\gamma_{i,j}} \geq \beta \tag{4}$$

This models a situation where a minimum signal-to-interference ratio (SIR) of β is necessary for successful receptions, the ambient noise power level is N, and signal power decays with distance r as $\frac{1}{r^\kappa}$. We will suppose that $\kappa > 2$, which is the usual model outside a small neighborhood of the transmitter.

3) *The Throughput Capacity of Random Networks:* The notion of throughput is defined in the usual manner as the time average of the number of bits per second that can be transmitted by every node to its destination.

III. Impact on Number of Simultaneous Source-Receiver Pairs

Theorem III.1. The simultaneous S-R pair number N_{S-R} is bounded as follows:

$$N_{S-R} \leq \frac{n+m}{2} \tag{5}$$

Proof: Assume at time t there are totally n_1 ($n_1 \leq m$) static nodes each communicating with a mobile relay node, and other static nodes, totally n_2 communicating with their static neighbors. We define a relay-communicating ratio $\theta \in (0,\ 1)$. Then $n_1 = \theta n$ and $n_2 = (1-\theta)n$. The total number of S-R pairs can thus be calculated as

$$\begin{aligned} N_{S-R} &= n_1 + \frac{n_2}{2} \\ &= \theta n + \frac{(1-\theta)n}{2} \\ &= \frac{1}{2}(1+\theta)n \\ &= \frac{n}{2} + \frac{n_1}{2} \\ &\leq \frac{n+m}{2}. \end{aligned}$$

■

The number of S-R pairs reaches the upper bound mentioned in Theorem III.1 when m is equal to θn. Hence, the relay-communicating ratio θ is $\frac{m}{n}$, which means that there are $n_1 = m$ static nodes communicating with relay nodes simultaneously. This reveals the fact that the more relay nodes involved in communication simultaneously, the larger number of S-R pairs supported by ad hoc network. Although intuitively, this result presents part of explanation for the impact of mobile relay nodes on static ad hoc network.

IV. Network Communication Using only Relays

Here we make a little change to the scheduling policy π presented by Matthias Grossglauser and David N. C. Tse [3] to fit the new model. For ease of mathamtical deduction, here we omit direct transmission between two static nodes and only transmission from one static node to one mobile relay node or from one mobile relay node to one static node is permitted under this scheduling policy. For each time instant t, the scheduling policy randomly choose n_S nodes to form the sender set $\mathcal{S}$. Each sender node in $\mathcal{S}$ has a randomly chosen intended destination toward which it transmits packet. All mobile relay nodes form the receiver set $\mathcal{R}$, which receives packets from nodes in $\mathcal{S}$. Each sender node in $\mathcal{S}$ chooses to transmit packet to the nearest node among all nodes in R. Whether a transmission is successful is dependent on the ratio of power generated by sender and interference generated by other senders. Denote by N_t the number of successful transmission.

Theorem IV.1. For the scheduling policy π, the expected number $E[N_t]$ of feasible sender-receiver pairs is $\Theta(n)$, i.e.,

$$\lim_{n\to\infty} \frac{E\left[N_t\right]}{n} = \phi > 0 \tag{6}$$

Furthermore, for two arbitrary nodes i and j, the probability that (i,j) is scheduled as a sender-receiver pair is $\Theta\left(\frac{1}{m}\right)$.

Proof: Here we review the proof of Theorem III.4 in [3] with a different model. We consider a fixed time t. Let $U_1, \cdots, U_{n_S}$ be the random positions of the senders in $\mathcal{S}$. Let $V_1, \cdots, V_{n_R}$ be the positions of nodes in the receiver set $\mathcal{R}$. These random variables are i.i.d. uniformly distributed on the open disk of unit area. For each node $s \in \mathcal{S}$, let its intended receiver $r(s) \in \mathcal{R}$ be the relay node that is nearest to s among all nodes in $\mathcal{R}$. Since the number of relay nodes in one ad hoc network is m, the number of receivers n_R available is m.

We now analyze the probability of successful transmission for each chosen sender-receiver pair. By symmetry, we can just focus on one such pair, say $(1, r(1))$. The event of successful transmission depends on the positions $U_1, \cdots, U_{n_S}$ and $V_1, \cdots, V_m$. Let Q_i be the received power from sender node i at receiver node $r(1)$, and

$$Q_i = \frac{P}{|U_i - V_{r(1)}|^\kappa}.$$

The node $r(1)$ satisfies

$$r(1) = \arg\min_j |U_1 - V_j|.$$

The total interference at node $r(1)$ is given by $I = \sum_{i\neq 1} Q_i$. The SIR for the transmission from sender 1 at receiver $r(1)$ is given by

$$SIR = \frac{Q_1}{N_0 + I}.$$

We now analyze the asymptotics of Q_1 and I as $n \to \infty$. Now

$$Q_1 = \max_{j=1,\cdots,m} Z_j$$

where $Z_j = \frac{P}{|U_1 - V_j|^\kappa}$. Let us first condition on $U_1 = u$ for some u in the open disk. A disk centered at u and of radius $r < (\pi^{-1/2} - |u|)$ lies entirely inside the unit disk. Then, for every $z > r^{-\kappa}$ and for all j, we have

$$\begin{aligned} Pr\left\{Z_j > z | U_i = u\right\} &= Pr\left\{|V_j - u| < z^{-\frac{1}{\kappa}}\right\} \\ &= \pi z^{-\frac{2}{\kappa}}. \end{aligned} \tag{7}$$

Conditional on $U_1 = u$, the random variables Z_i's are i.i.d. By a standard result on the asymptotic distribution of extremum of i.i.d. random variables [1], the extremum Q_1 of m i.i.d. random variables whose cdf satisfies

$$\lim_{m\to\infty} \frac{1 - F_Z(x)}{1 - F_Z(kx)} = k^b \tag{8}$$

satisfies

$$\lim_{m\to\infty} Pr\left\{Q_1 \le a_m x\right\} = \exp(-x^{-b}) \tag{9}$$

where a_m is given by $a_m = F_Z^{-1}(1 - \frac{1}{m}) = (\pi m)^{\frac{\kappa}{2}}$. Comparing Equation 7 with Equation 9, the asymptotic distribution of Q_1 is

$$\lim_{m\to\infty} Pr\{Q_1 < a_m x | U_1 = u\} = F_{Q_a^*}(x) \tag{10}$$

where Q_a^* has a cdf

$$F_{Q_a^*} = \begin{cases} \exp(-x^{-\frac{2}{\kappa}}), & x \ge 0 \\ 0, & x < 0. \end{cases} \tag{11}$$

Then, for each $x > 0$, we have

$$\begin{aligned} &\lim_{m\to\infty} Pr\left\{Q_1 < a_m x\right\} \\ &= \lim_{m\to\infty} \int_{u\in D} Pr\left\{Q_1 < a_m x | U_1 = u\right\} du \\ &= \int_{u\in D} \lim_{m\to\infty} Pr\left\{Q_1 < a_m x | U_1 = u\right\} du \\ &= \int_{u\in D} F_{Q_\alpha^*}(x)\, du = F_{Q_\alpha^*}(x) \end{aligned} \tag{12}$$

Since $F_{\frac{Q_1}{a_m}}(x) = \lim_{m\to\infty} Pr\{Q_1 < a_m x\}$, we conclude that random variable $\frac{Q_1}{a_m}$ coverges to random variable Q_a^*. That is

$$(\pi m)^{-\frac{\kappa}{2}} Q_1 \overset{m\to\infty}{\longrightarrow} Q_\alpha^*. \tag{13}$$

Same as the result in [3], conditional on $V_{r(1)} = u$, the interference $I = \sum_{i=2}^{n_S}$ satifies

$$\begin{aligned} \left[\pi\Gamma\left(1 - \frac{2}{\kappa}\right) n_S\right]^{-\frac{\kappa}{2}} I &= \left[\pi\Gamma\left(1 - \frac{2}{\kappa}\right) \theta n\right]^{-\frac{\kappa}{2}} \\ &\overset{n\to\infty}{\longrightarrow} I_\alpha^* \end{aligned} \tag{14}$$

As claimed by Grossglauser [3], the signal power and the total interference are asymptotically independent. Hence, combining this claimation with Equation 13 and Equation 14, the probability of successful transmission from sender 1 to receiver $r(1)$ is

$$\begin{aligned} Pr\{SIR > \beta\} &= Pr\left\{\frac{Q_1}{N_0 + I} > \beta\right\} \\ &\overset{n\to\infty}{\underset{m\to\infty}{\longrightarrow}} Pr\left\{\frac{Q_\alpha^*}{I_\alpha^*} > \beta^*\right\} > 0, \end{aligned} \tag{15}$$

where β^* is given by

$$\begin{aligned} \beta^* &= \frac{(\pi m)^{-\frac{\kappa}{2}}}{\left[\pi\Gamma\left(1 - \frac{2}{\kappa}\right)\theta n\right]^{-\frac{\kappa}{2}}} \beta \\ &= \beta \left[\theta\psi\Gamma\left(1 - \frac{2}{\kappa}\right)\right]^{\frac{\kappa}{2}}, \end{aligned} \tag{16}$$

where $\psi \in (1, +\infty)$ is the node-to-relay ratio $\frac{n}{m}$.

The expected number of successful transmission at one time instant t is therefore

$$E[N_t] = \theta n Pr\left\{SIR > \beta\right\}, \tag{17}$$

$$\phi = \theta Pr\left\{SIR > \beta\right\}. \tag{18}$$

Furthermore, since scheduling policy π depends only on locations of sender nodes and receiver nodes, and since locations of nodes $\{X_i\}$ and $\{Y_i\}$ are i.i.d, the probability of successful transmission between any specific sender-receiver pair is equal, and thus, $\Theta\left(\frac{1}{n}\right)$. ■

Since there exist totally $m = \frac{n}{\psi}$ mobile relays in the network, the probability that one sender can transmit to a mobile relay at time slot t is

$$\begin{aligned} &\text{Prob}\left(\text{one transmits to a mobile relay at time slot } t\right) \\ &= E\left[N_t\right] \times \frac{n}{\psi} = \Theta\left(\frac{1}{\psi}\right) = \Theta\left(1\right), \end{aligned} \tag{19}$$

which is the same as Grossglauser and Tse's result [3].

V. Network with Finite Mobile Relay Nodes

Although the previous result provides us with an optimistic vision, the assumption that the number of mobile relay nodes m is of the same order of static nodes n is too strong. Under many circumstances, only finite mobile relay nodes are permitted in one network and thus an overall capacity $\Theta\left(1\right)$ is unreachable.

It is proved by Gupta and Kumar [2] that under a Protocol Model of noninterference, the capacity of wireless networks with n randomly located nodes each capable of transmitting at W bits per second and employing a common range, and each with randomly chosen and therefore likely far away destination, is $\Theta\left(\frac{W}{\sqrt{n\log n}}\right)$. Although the mathematical induction was done under the structure of a sphere $\mathcal{S}^2$ of unit space, Gupta and Kumar proved that this result is also applicable for nodes located on planar disk.

In this section, all nodes are deployed on a planar disk of unit area. n static nodes are randomly located, i.e., independently and uniformly distributed, on the disk, and m mobile relay nodes are added into the network, of which the process $\{X_i\left(t\right)\}$ is assumed to be stationary and ergodic with stationary distributiion uniform on the disk.

Only finite mobile relay nodes are added in the ad hoc network. The number of mobile relay nodes is denoted as m, and unlike what we assume in Section IV, m is a variable far smaller than the number of static nodes n. The node-to-relay ratio, no longer a constant, is denoted as $\psi\left(n\right) = \frac{n}{m} \in (1, +\infty)$.

A. Scheduling Policy

The scheduling policy π is as follows.

1) We fix a parameter *sender density* $\theta \in (0,1)$, and the number of senders n_S can thus be exhibited as $n_S = \theta n$. All sender nodes form the sender set S. For each time slot t, one node from all other static nodes is chosen as destination of one given sender node.
2) Nodes not included in the sender set $\mathcal{S}$, both static nodes and mobile relay nodes, are included in the receiver set $\mathcal{R}$. For each time slot t, each sender node in $\mathcal{S}$ transmits packets to its nearest neighbor among nodes in $\mathcal{R}$. Both static neighbors and mobile relay nodes can be chosen as receiver. It should be mentioned that even if a static node is chosen as receiver at one particular time slot t_1 of one packet p_i, it does not necessarily follow that packet p_i will be transmitted through a multihop method. It is permitted that static nodes chosen as receiver at one time slot of packet p_i can transmit this packet to one mobile relay node in the next time slot. The number of static nodes in $\mathcal{R}$ is denoted as n_R. Obviously, we have $n_R = (1-\theta)\, n$.
3) On accordance to the physical model of a random model, all sender nodes each time transmits to its chosen receiver with unit power $(P_i = 1)$.
4) For each time slot t, we retain those successful sender-receiver pairs (S-R pairs), and the number of these successful S-R pairs is denoted as N_t.

B. Probability of Transmission to a Mobile Relay Node

In order to combine multi-hop transmission through static nodes and two-hop transmission through one mobile relay node, we have to first study the probability that one given source node chooses one mobile relay node as its receiver at a particular time slot t. According to scheduling policy π, at each time slot t, the source nodes choose the nearest node to them as their receiver, with no regard of whether it is a mobile relay node or a static node. Therefore, the probability that one given source node chooses one mobile relay node at a particular time slot t is equal to the probability that one mobile relay node is the nearest node to the given source node. In the following part, we use A_i to denote the proposition that one mobile relay node is the nearest node to the given souce node X_i.

Since mobile relay nodes are distributed uniformly on the disk, it follows that $\text{Prob}\,(A_i)$ is equal to the area S_{circle} of the circle with center at X_i and radius of $|X_i - X_{neib(i)}|$, where $X_{neib(i)}$ denotes the nearest static neighbor node of node X_i. Since n static nodes are uniformly distributed on a planar disk of unit area, distance between neighbors is $d = \Theta\left(\frac{1}{\sqrt{n}}\right)$. Thus the area of the circle is $S_{circle} = \Theta\left(\frac{1}{n}\right)$. Therefore, we have

$$\text{Prob}\,(A_i) = \Theta\left(\frac{1}{n}\right). \tag{20}$$

Since there are totally m uniformly distributed mobile relay nodes in the network, the probability of transmission to a mobile relay node is thus

$$\begin{aligned}&\text{Prob}\,(\text{one given source node chooses a mobile}\\&\quad\text{relay node as receiver}) = \Theta\left(\frac{m}{n}\right),\end{aligned} \tag{21}$$

and the probability that a static neighbor node is chosen as receiver is

$$\begin{aligned}&\text{Prob}\,(\text{one given source node chooses a static}\\&\quad\text{neighbor node as receiver}) = \Theta\left(1-\frac{m}{n}\right).\end{aligned} \tag{22}$$

In the following part, we denote probability in Equation 21 as $\text{Prob}\,(A)$ and probability in Equation 22 as $\text{Prob}\,(B)$.

C. The Throughput at Each Node

It is proved in Section IV that under the assumption that each source node can find one mobile relay node at any time slot the throughput per node is $\lambda_{two-hop}\,(n) = \Theta\,(1)$, and Gupta and Kumar proved in [2] that the throughput per node of a static network with transmission through multihop using static nodes as relays is $\lambda_{multihop}\,(n) = \Theta\left(\frac{1}{\sqrt{n \log n}}\right)$. Combining these two results, as well as Equation 21 and Equation 22, the throughput per node can be calculated.

At each time slot t, each sender node chooses one mobile relay node as receiver with probability $\text{Prob}\,(A)$, or one static neighbor with probability $\text{Prob}\,(B)$. Then, with throughput provided by two-hop transmission and multi-hop transmission, the overall throughput per node can be calculated. Since here the two-hop transmission throughput and multi-hop transmission throughput are all on the condition that corresponding relays are chosen, these two throughputs are conditional throughput. We use $\lambda_{two-hop}\,(n|A)$ to denote conditional throughput of two-hop transmission and $\lambda_{multi-hop}\,(n|B)$ to denote conditional throughput of multi-hop transmission.

Since the conditional throughput can be interpreted as throughput with assumption that corresponding condition happens with probability 1, we can simply apply results of Gupta and Kumar [2] and in section IV. That is

$$\lambda_{two-hop}\,(n|A) = \Theta\,(1) \tag{23}$$

$$\lambda_{multi-hop}\,(n|B) = \Theta\left(\frac{1}{\sqrt{n \log n}}\right). \tag{24}$$

Theorem V.1. The throughput per node is

$$\lambda\,(n) = \Theta\left(\frac{1}{\psi\,(n)} + \frac{1}{\sqrt{n \log n}} - \frac{1}{\psi\,(n)}\frac{1}{\sqrt{n \log n}}\right).$$

Proof: The throughput per node can viewed as a combination of two possible transmission method, each calculated as a conditional probability.

$$\begin{aligned}\lambda(n) &= \text{Prob}(A)\,\lambda_{two-hop}(n|A) \\ &+ \text{Prob}(B)\,\lambda_{multi-hop}(n|B) \\ &= \Theta\left(\frac{m}{n}\right)\Theta(1) + \Theta\left(\frac{n-m}{n}\right)\Theta\left(\frac{1}{\sqrt{n\log n}}\right) \\ &= \Theta\left(\frac{m}{n}\right) + \Theta\left(\frac{n-m}{n}\frac{1}{\sqrt{n\log n}}\right) \\ &= \Theta\left(\frac{m}{n} + \frac{1}{\sqrt{n\log n}} - \frac{m}{n}\frac{1}{\sqrt{n\log n}}\right) \\ &= \Theta\left(\frac{1}{\psi(n)} + \frac{1}{\sqrt{n\log n}} - \frac{1}{\psi(n)}\frac{1}{\sqrt{n\log n}}\right). \quad (25)\end{aligned}$$

■

This result can be viewed as a generalized result combining network with infinite relay nodes and network with finite relay nodes. When the number of relay nodes is of same order with the number of static nodes, that is, $\frac{n}{m} = \psi$, where ψ is a constant, Equation 25 converges to $\Theta(1)$, which is Grossglauser and Tse's result, and when the number of relay nodes is far smaller than the number of static nodes, that is, $\frac{n}{m} = \psi(n) \overset{n\to\infty}{\longrightarrow} 0$, Equation 25 converges to $\Theta\left(\frac{1}{\sqrt{n\log n}}\right)$, which is Gupta and Kumar's result.

VI. Conclusion

In this paper we study static network with mobile relays added in order to compensate the decrease of throughput with the increase of number of nodes n. We study two cases, the first of which is static network with infinite mobile relay nodes, and the second of which is static network with finite mobile relay nodes. We show that with infinite mobile relay nodes, the throughput can reach the upper bound of ad hoc network. Our work provides a combination of Gupta and Kumar's multi-hop network and Grossglauser and Tse's random work network.

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