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Compressive detection with sparse random projections

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Abstract: This paper addresses the problem of sparse signal detection based on compressive sensing (CS) framework. In order to reduce the computational complexity of detection, we propose a sparse compressive detection approach by substituting sparse random projection matrix for conventional dense matrix, and develop a theoretical model that exactly characterizes the relationship of the detection probability with the number of measurements, the signal-to-noise ratio (SNR), as well as the degree of sparsity. Simulation results show that the performance of the proposed detector is comparable to the conventional CS detector in which Gaussian dense projections are employed.

Keywords: compressive sensing, signal detection, sparse random projections, detection probability
Classification: Fundamental Theories for Communications

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1 Introduction

The theory of compressive sensing (CS) [1] shows that a compressible or sparse signal can be reconstructed from far fewer samples than traditional Nyquist sampling. Due to this advantage, CS is potentially attractive for many applications such as signal acquisition, image compression, and communications. In this paper, we focus on the design of compressive detector with sparse random projections.

CS based detection was first studied by Duarte et al [2]. They solved signal detection problems without reconstructing the signals involved, which significantly reduces the number of measurements required. Davenport et al [3] expanded this work on detection and estimation, and bounded the performance of the detector in which the projection matrix is a random orthoprojector. Similarly, the authors in [4] proposed compressive image classification which operates directly on the compressive measurements, and analyzed the impacts of the noise level and the number of measurements on the classification performance.

In order to reduce the number of measurements, the authors in [5] and [6] developed subspace compressive detectors. They are designed to concentratively capture signal energy in a low dimension subspace where the sparse signal resides in, thus leading to fewer samples needed for a given detection performance than that of [3]. However, such methods require not only signal training for the information of the signal subspace, but adaptive detectors tailored to particular signal structure.

The projection systems used in the aforementioned works are typically based on some kinds of "dense" matrices, e.g., Gaussian or Bernoulli matrices. Dense matrices and their operations (e.g. matrix multiplication) may lead to difficult or costly hardware implementations. For large scale problems, it becomes intractable in practice.

To cope with this problem, Yin et al [8] considered easily realized Toeplitz and circulant matrices, and proposed fast reconstruction algorithms for incomplete Toeplitz and circulant measurements. Rauhut et al [7] introduced two types of structured random matrices, and the corresponding signal recovery using ℓ_1-minimization. Unfortunately, these studies are limited on signal reconstruction. How to apply them in specific detection scenarios without signal recovery remains vastly unexplored.

This paper is motivated to design a simple and practicable CS framework for a known sparse signal detection. Our main contributions are twofold.

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First, we propose a sparse compressive detection approach. It reduces the computational complexity of detection by substituting sparse random projection matrix for conventional dense matrix. Then we provide a decision strategy with probabilistic performance guarantees for detection problems. Second, a theoretical model is developed that exactly characterizes the relationship of the detection probability with the number of measurements, the signal-to-noise ratio (SNR), as well as the degree of sparsity. This model provides a practical guideline on choosing some key parameters when the proposed scheme is applied in a real system.

2 Methodology and modeling

2.1 Compressive sensing background

Consider a real-valued signal $\mathbf{x} \in \mathbb{R}^N$. Assume that the basis $\Psi = \{\psi_1, ..., \psi_N\}$ provides a K-sparse representation of $\mathbf{x}$. That is $\mathbf{x} = \Psi\theta$, where the coefficient vector θ is an $N \times 1$ column vector with K nonzero elements. The CS theory states that it is possible to reconstruct $\mathbf{x}$ from measurements

$$\mathbf{y} = \Phi\mathbf{x} = \Phi\Psi\theta \tag{1}$$

where Φ is an $M \times N$ projection or measurement matrix with $M \ll N$, $\mathbf{y}$ is an $M \times 1$ vector representing the random linear projections of $\mathbf{x}$ onto the projection matrix Φ.

Since $M \ll N$, the recovery of the original signal $\mathbf{x}$ from the measurements $\mathbf{y}$ is generally ill-conditioned. However, the additional assumption that $\mathbf{x}$ is K-sparse guarantees recovery possible and practical. In particular, the recovery of $\mathbf{x}$ can be achieved by exploring the sparse expression of $\mathbf{x}$, i.e., seeking the sparsest coefficient vector among all possible θ that satisfies $\mathbf{y} = \Phi\Psi\theta$. Theoretically, perfect reconstruction can be achieved using ℓ_1 minimization

$$\hat{\theta} = \arg\min \|\theta\|_1 \;, \quad \text{s.t.} \quad \mathbf{y} = \Phi\Psi\theta \tag{2}$$

2.2 Sparse compressive detection

When CS theory is applied in signal detection, the target becomes to distinguish between the following two hypotheses $\mathcal{H}_0$ and $\mathcal{H}_1$:

$$\begin{aligned} &\mathcal{H}_0 : \mathbf{y} = \Phi n \\ &\mathcal{H}_1 : \mathbf{y} = \Phi(\mathbf{x} + \mathbf{n}) \end{aligned} \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^N$ is a known signal, Φ is an $M \times N$ projection matrix, and $\mathbf{n} \sim \mathcal{N}(0, \sigma^2 I_N)$ is Gaussian noise.

Unlike dense projections used in the conventional CS, we construct sparse random projection matrix Φ containing i.i.d. entries [9]:

$$\phi_{ij} = \sqrt{s}\begin{cases} +1 & \frac{1}{2s} \\ 0 & 1 - \frac{1}{s} \\ -1 & \frac{1}{2s} \end{cases} \tag{4}$$

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In this case, the sufficient statistic is defined as:

$$t = \mathbf{y}^T(\Phi\Phi^T)^{-1}\Phi\mathbf{x} \tag{5}$$

where we have

$$\begin{aligned} t &\backsim \mathcal{N}(0, \sigma^2\mathbf{x}^T\Phi^T(\Phi\Phi^T)^{-1}\Phi\mathbf{x}) \quad \text{under } \mathcal{H}_0 \\ t &\backsim \mathcal{N}(\mathbf{x}^T\Phi^T(\Phi\Phi)^{-1}\mathbf{x}, \sigma^2\mathbf{x}^T\Phi^T(\Phi\Phi^T)^{-1}\Phi\mathbf{x}) \quad \text{under } \mathcal{H}_1 \end{aligned} \tag{6}$$

Given an appropriate threshold γ, the performance of the detection system is given by [3]

$$P_D = P(t > \gamma|\mathcal{H}_1) = Q\left(\frac{\gamma - (\Phi\mathbf{x})^T(\Phi\Phi^T)^{-1}\Phi\mathbf{x}}{\sigma\sqrt{(\Phi\mathbf{x})^T(\Phi\Phi^T)^{-1}\Phi\mathbf{x}}}\right) \tag{7}$$

$$P_F = P(t > \gamma|\mathcal{H}_0) = Q\left(\frac{\gamma}{\sigma\sqrt{(\Phi\mathbf{x})^T(\Phi\Phi^T)^{-1}\Phi\mathbf{x}}}\right) \tag{8}$$

where P_D is the probability of choosing $\mathcal{H}_1$ when $\mathcal{H}_1$ is true (i.e. the probability of detection), and P_F is the probability of choosing $\mathcal{H}_1$ when $\mathcal{H}_0$ is true (i.e. the probability of false alarm).

Keeping P_F under a certain value α, the performance of the proposed detector can be approximated as:

$$P_D(\alpha) = Q\Big(Q^{-1}(\alpha) - \frac{\sqrt{(\Phi\mathbf{x})^T(\Phi\Phi^T)^{-1}\Phi\mathbf{x}}}{\sigma}\Big) \tag{9}$$

Lemma 1:[9] Consider a vector $\mathbf{x} \in \mathbb{R}^N$ and a random matrix $\Phi \in \mathbb{R}^{M\times N}$ with entries ϕ_{ij} satisfying the conditions in Eq. (4). It is shown that

$$E[\|\Phi\mathbf{x}\|_2^2] = M\|\mathbf{x}\|_2^2 \tag{10}$$

$$Var[\|\Phi\mathbf{x}\|_2^2] = M\Big(2\|\mathbf{x}\|_2^4 + (s-3)\sum_{i=1}^{N} x_i^4\Big) \tag{11}$$

Theorem 1: Let Φ be an $M \times N$ matrix satisfying the conditions in Eq. (4). For any $\mathbf{x} \in \mathbb{R}^N$ and $\epsilon \in (0,1)$, with probability at least $1 - \frac{1}{\epsilon^2}\frac{1}{M\|\mathbf{x}\|_2^4}(2\|\mathbf{x}\|_2^4 + (s-3)\sum_{i=1}^{N} x_i^4)$, P_D satisfies:

$$\begin{aligned} &Q\Big(Q^{-1}(\alpha) - \sqrt{1-\epsilon}\sqrt{\frac{M}{N}}\sqrt{SNR}\Big) < P_D(\alpha) \\ &< Q\Big(Q^{-1}(\alpha) - \sqrt{1+\epsilon}\sqrt{\frac{M}{N}}\sqrt{SNR}\Big) \end{aligned} \tag{12}$$

where $\sqrt{SNR} \triangleq \frac{\|\mathbf{x}\|_2}{\sigma}$.

Proof: Since Φ satisfies the conditions in Eq. (4), we have

$$E[\phi_{ij}] = 0 \tag{13}$$

$$E[\phi_{ij}^2] = \frac{s}{2s} + \frac{s}{2s} = 1 \tag{14}$$

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Let $\Phi\Phi^T \triangleq P$, then $P \in \mathcal{R}^{N\times N}$ and the entries are denoted as $P_{ij} = \sum_{k=1}^{N} \phi_{ik}\phi_{jk}$, with

$$E[P_{ij}] = E\Big[\sum_{k=1}^{N} \phi_{ik}\phi_{jk}\Big] = \sum_{k=1}^{N} E[\phi_{ik}]E[\phi_{jk}] = 0 \quad (i \neq j)$$
$$E[P_{ij}] = E\Big[\sum_{k=1}^{N} \phi_{ik}^2\Big] = \sum_{k=1}^{N} E[\phi_{ik}^2] = N \quad (i = j) \tag{15}$$

Ignoring the diagonal entries, we have

$$P \approx NI_{M\times M} \tag{16}$$

Then $(\Phi\mathbf{x})^T(\Phi\Phi^T)^{-1}\Phi x$ can be approximated as:

$$(\Phi\mathbf{x})^T(\Phi\Phi^T)^{-1}\Phi\mathbf{x} \approx \frac{1}{N}\|\Phi\mathbf{x}\|_2^2 \tag{17}$$

Thus, we have

$$P_D(\alpha) \approx Q\Big(Q^{-1}(\alpha) - \sqrt{\frac{1}{N}}\frac{\|\Phi\mathbf{x}\|_2}{\sigma}\Big) \tag{18}$$

By Lemma 1 and Chebyshev inequality, we have

$$\begin{aligned}&\mathbf{Prob}\Big(M\|\mathbf{x}\|_2^2 - \epsilon < \|\Phi\mathbf{x}\|_2^2 < M\|\mathbf{x}\|_2^2 + \epsilon\Big)\\ &\geq 1 - Var(\|\Phi\mathbf{x}\|_2^2)\\ &= 1 - \frac{1}{\epsilon^2}M\Big(2\|\mathbf{x}\|_2^4 + (s-3)\sum_{i=1}^{N} x_i^4\Big)\end{aligned} \tag{19}$$

Then

$$\begin{aligned}&\mathbf{Prob}\Big(1-\epsilon < \frac{1}{M}\frac{\|\Phi\mathbf{x}\|_2^2}{\|\mathbf{x}\|_2^2} < 1+\epsilon\Big)\\ &\geq 1 - \frac{1}{\epsilon^2}\frac{1}{M\|\mathbf{x}\|_2^4}\Big(2\|\mathbf{x}\|_2^4 + (s-3)\sum_{i=1}^{N} x_i^4\Big)\end{aligned} \tag{20}$$

$$\begin{aligned}&\mathbf{Prob}\Big(\sqrt{1-\epsilon} < \frac{1}{\sqrt{M}}\frac{\|\Phi\mathbf{x}\|_2}{\|\mathbf{x}\|_2} < \sqrt{1+\epsilon}\Big)\\ &\geq 1 - \frac{1}{\epsilon^2}\frac{1}{M\|\mathbf{x}\|_2^4}\Big(2\|\mathbf{x}\|_2^4 + (s-3)\sum_{i=1}^{N} x_i^4\Big)\end{aligned} \tag{21}$$

Combining with Eq. (18), we can rewrite Eq. (9) as

$$\begin{aligned}&\mathbf{Prob}\Big(Q\Big(Q^{-1}(\alpha) - \sqrt{1-\epsilon}\sqrt{\frac{M}{N}}\sqrt{SNR}\Big) < P_D(\alpha)\\ &< Q\Big(Q^{-1}(\alpha) - \sqrt{1+\epsilon}\sqrt{\frac{M}{N}}\sqrt{SNR}\Big)\Big)\\ &\geq 1 - \frac{1}{\epsilon^2}\frac{1}{M\|\mathbf{x}\|_2^4}\Big(2\|\mathbf{x}\|_2^4 + (s-3)\sum_{i=1}^{N} x_i^4\Big)\end{aligned} \tag{22}$$

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It is observed that the probability of detection P_D depends on the number of measurements, the SNR of the system, and the sparsity degree s of the projection matrix. When the SNR and the sparsity degree s is pre-defined, the performance of the proposed detector can be uniquely determined by the number of measurements.

From Theorem 1, we can also achieve the upper bound and the lower bound of P_D.

$$P_D(\alpha)_{\text{lower}} = Q\Big(Q^{-1}(\alpha) - \sqrt{1-\epsilon}\sqrt{\frac{M}{N}}\sqrt{SNR}\Big) \tag{23}$$

$$P_D(\alpha)_{\text{upper}} = Q\Big(Q^{-1}(\alpha) - \sqrt{1+\epsilon}\sqrt{\frac{M}{N}}\sqrt{SNR}\Big) \tag{24}$$

It tells that with the probability no less than $1 - \frac{1}{\epsilon^2}\frac{1}{M\|\mathbf{x}\|_2^4}(2\|\mathbf{x}\|_2^4 + (s-3)\sum_{i=1}^{N} x_i^4)$, P_D varies between the lower bound and the upper bound, with a mean approximating $Q(Q^{-1}(\alpha) - \sqrt{M/N}\sqrt{SNR})$.

3 Simulation results

In this section, we present some numerical results to demonstrate the performance of the proposed detection approach.

Fig. 1 shows the impacts of the number of measurements M (or sampling ratio M/N) on the system performance at different SNR levels. Here we fix $P_F = 0.01$ and vary the SNR from 10 dB to 25 dB. As can be seen from this figure, the detection probability P_D increase with both the SNR and the ratio M/N. It is clear that the detection performance can be improved under better SNR conditions or when performing more measurements. Specifically, when the SNR is at a relatively high level (20 dB in this case) with $M/N = 0.5$, the detection probability of the detector can approach 1.

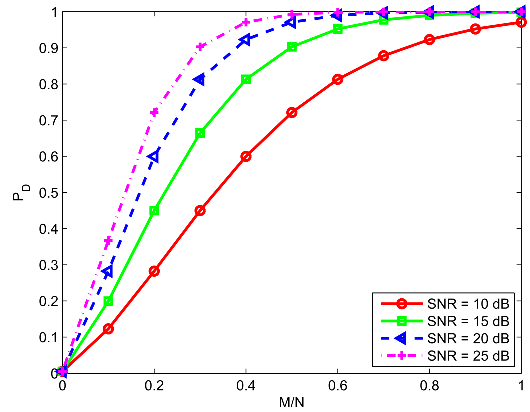

Fig. 1. Imacts of M on P_D at different SNR levels (P_F=0.01)

Fig. 2 plots receiver operating characteristic (ROC) curves for different sampling ratio with a fixed SNR=20 dB. It can be found that when we increase the sampling ratio, the system performance can be significantly improved. In a case of $M = 0.4N$, the detector can achieve $P_D = 1$ with the probability of false alarm P_F equal to 0.1.

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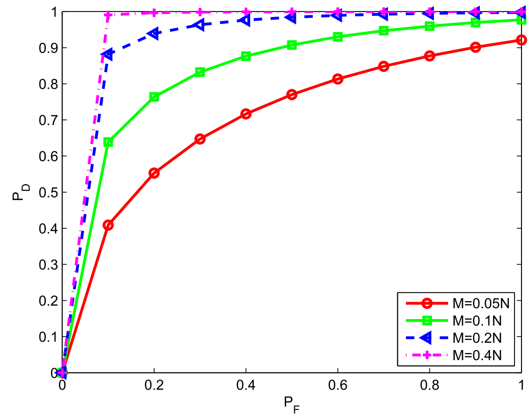

Fig. 2. ROC curves for different sampling ratio M/N

Eq. (23) and (24) show that with the decrease of ϵ, the lower bound curve and the upper bound curve of the proposed detector gets closer to each other. It is observed from Fig. 3 that when $\epsilon = 0.1$, the lower bound curve and the upper bound curve of the proposed detector not only coincide with each other, but coincide with ROC curve of the conventional CS detector. It means that with a relatively small ϵ, the performance of the proposed detector can approximate to the conventional CS detector, while the former has a lower complexity.

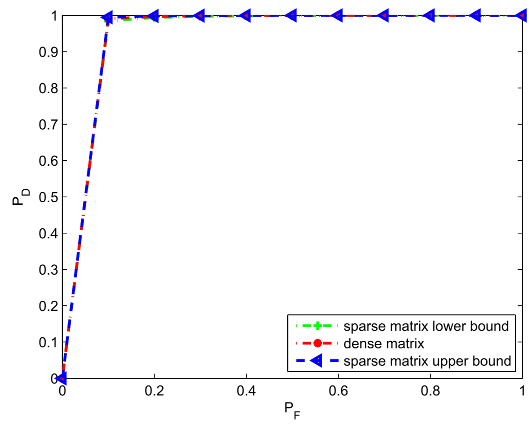

Fig. 3. ROC comparison (SNR=20 dB, M=0.4N, $\epsilon = 0.1$)

4 Conclusion

This paper studied sparse signal detection with compressive sensing. To reduce computation cost, we utilized sparse random projections instead of Gaussian matrix to generate measurements, and made detection decision directly on compressive measurements. The performance of the proposed detector was theoretically modeled. Simulation results showed that our approach can achieve a similar performance with the Gaussian matrix detector.

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